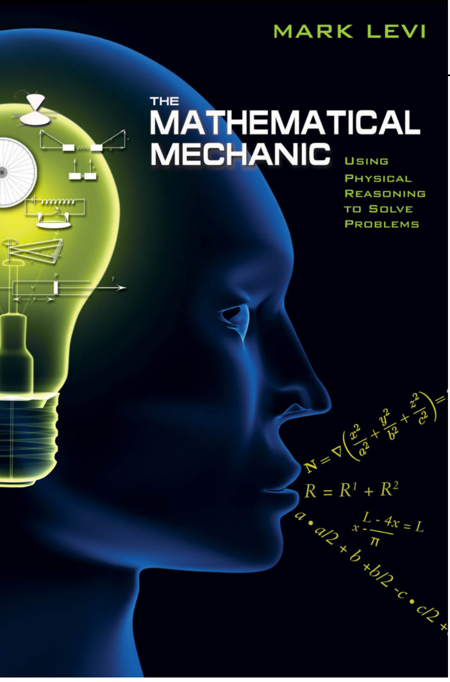
MARK LEVI
THE
MATHEMATICAL
MECHANIC
Using
Physical
Reasoning
to Solve
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THE MATHEMATICAL MECHANIC

USING PHYSICAL REASONING TO SOLVE PROBLEMS

PRINCETON UNIVERSITY PRESS PRINCETON AND OXFORD

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 6 Oxford Street,
Woodstock, Oxfordshire OX20 1TW

press.princeton.edu

Eighth printing, and first paperback printing, 2012
Paperback ISBN 978-0-691-15456-5

The Library of Congress has cataloged the cloth edition of this book as follows
Levi, Mark, 1951–
The mathematical mechanic: using physical reasoning to
solve problems / Mark Levi.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-691-14020-9 (cloth : alk. paper)
1. Problem solving. 2. Mathematical physics. I. Title.
QA63.L48 2009
510–dc22 2009004861

British Library Cataloging-in-Publication Data is available

This book has been composed in Times

Printed on acid-free paper. ∞

Typeset by S R Nova Pvt Ltd, Bangalore, India

Printed in the United States of America

10 9 8

Contents

THE MATHEMATICAL MECHANIC

1

INTRODUCTION

> IT SO HAPPENS THAT ONE OF THE GREATEST MATHEMATICAL DISCOVERIES OF ALL TIMES WAS GUIDED BY PHYSICAL INTUITION.
>
> —GEORGE POLYA, ON ARCHIMEDES' DISCOVERY OF INTEGRAL CALCULUS

1.1 Math versus Physics

Back in the Soviet Union in the early 1970s, our undergraduate class—about forty mathematics and physics sophomores—was drafted for a summer job in the countryside. Our job included mixing concrete and constructing silos on one of the collective farms. My friend Anatole and I were detailed to shovel gravel. The finals were just behind us and we felt free (as free as one could feel in the circumstances). Anatole's major was physics; mine was mathematics. Like the fans of two rival teams, each of us tried to convince the other that his field was superior. Anatole said bluntly that mathematics is a servant of physics. I countered that mathematics can exist without physics and not the other way around. Theorems, I added, are permanent. Physical theories come and go. Although I did not volunteer this information to Anatole, my own reason for majoring in mathematics was to learn the main *tool* of physics—the field which I had planned to eventually pursue. In fact, the summer between high school and college I had bumped into my high school physics teacher, who asked me about my plans for the Fall. "Starting on my math major," I said. "What? Mathematics? You are nuts!" was his reply. I took it as a compliment (perhaps proving his point).

1.2 What This Book Is About

This is not "one of those big, fat paperbacks, intended to while away a monsoon or two, which, if thrown with a good overarm action, will bring a water buffalo to its knees" (Nancy Banks-Smith, a British television critic). With its small weight this book will not bring people to their knees, at least not by its *physical* impact. However, the book does exact revenge—or maybe just administers a pinprick—against the view that mathematics is a servant of physics. In this book physics is put to work for mathematics, proving to be a very efficient servant (with apologies to physicists). Physical ideas can be real eye-openers and can suggest a strikingly simplified solution to a mathematical problem. The two subjects are so intimately intertwined that both suffer if separated. An occasional role reversal can be very fruitful, as this book illustrates. It may be argued that the separation of the two subjects is artificial.[1]

Some history. The Physical approach to mathematics goes back at least to Archimedes (c. 287 BC – c. 212 BC), who proved his famous integral calculus theorem on the volumes of the cylinder, a sphere, and a cone using an imagined balancing scale. The sketch of this theorem was engraved on his tombstone. Archimedes' approach can be found in [P]. For Newton, the two subjects were one. The books [U] and [BB] present very nice physical solutions of mathematical problems. Many of fundamental mathematical discoveries (Hamilton, Riemann, Lagrange, Jacobi, Möbius, Grassmann, Poincaré) were guided by physical considerations.

Is there a general recipe to the physical approach? As with any tool—physical[2] or intellectual—this one sometimes works and sometimes does not. The main difficulty is to come up with a

[1] "Mathematics is the branch of theoretical physics where the experiments are cheap" (V. Arnold [ARN]). Not only are the experiments in this book cheap—they are even free, being the thought experiments (see, for instance, problems 2.2, 3.3, 3.13, and, in fact, most of the problems in this book).

[2] With apologies for the pun.

physical incarnation of the problem.[3] Some problems are well suited for this treatment, and some are not (naturally, this book includes only the former kind). Finding a physical interpretation of a particular problem is sometimes easy, and sometimes not; readers can form their own opinions by skimming through these pages.

One lesson a student can take from this book is that looking for a physical meaning in mathematics can pay off.

Mathematical rigor. Our physical arguments are not rigorous, as they stand. Rather, these arguments are sketches of rigorous proofs, expressed in physical terms. I translated these physical "proofs" into mathematical proofs only for a few selected problems. Doing so systematically would have turned this book into a "big, fat ...". I hope that the reader will see the pattern and, if interested, will be able to treat the cases I did not treat. Having made this disclaimer I feel less guilty about using the word "proof" throughout the text without quotation marks.

The main point here is that the physical argument can be a tool of discovery and of intuitive insight—the two steps preceding rigor. As Archimedes wrote, "For of course it is easier to establish a proof if one has in this way previously obtained a conception of the question, than for him to seek it without such a preliminary notion" ([ARC], p. 8).

An axiomatic approach. Instead of translating each physical "proof" into a rigorous proof, an interesting project would entail systematically developing "physical axioms"—a set of axioms equivalent to Euclidean geometry/calculus—and then repeating the proofs given here in the new setting.

One can imagine an extraterrestrial civilization that first developed mechanics as a rigorous and pure axiomatic subject. In this dual world, someone would have written a book on using geometry to prove mechanical theorems.

Perhaps the real lesson is that one should not focus solely on one or the other approach, but rather look at both sides of the coin. This

[3] It is a contrarian approach: normally one starts with a physical problem, and abstracts it to a mathematical one; here we go in the opposite direction.

book is a reaction to the prevalent neglect of the physical aspect of mathematics.

Some psychology. Physical solutions from this book can be translated into mathematical language. However, something would be lost in this translation. Mechanical intuition is a basic attribute of our intellect, as basic as our geometrical imagination, and not to use it is to neglect a powerful tool we possess. Mechanics is geometry with the emphasis on motion and touch. In the latter two respects, mechanics gives us an extra dimension of perception. It is this that allows us to view mathematics from a different angle, as described in this book.

There is a sad Darwinian principle at work. Physical reasoning was responsible for some fundamental mathematical discoveries, from Archimedes, to Riemann, to Poincaré, and up to the present day. As a subject develops, however, this heuristic reasoning becomes forgotten. As a result, students are often unaware of the intuitive foundations of subjects they study.

The intended audience. If you are interested in mathematics and physics you will, I hope, not toss this book away.

This book may interest anyone who thinks it is fascinating that

- The Pythagorean theorem can be explained by the law of conservation of energy.
- Flipping a switch in a simple circuit proves the inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$.
- Some difficult calculus problems can be solved easily with no calculus.
- Examining the motion of a bike wheel proves the Gauss-Bonnet formula (no prior exposure is assumed; all the background is provided).
- Both the Cauchy integral formula and the Riemann mapping theorem (both explained in the appropriate section) become intuitively obvious by observing fluid motion.

This book should appeal to anyone curious about geometry or mechanics, or to many people who are not interested in mathematics because they find it dry or boring.

Uses in courses. Besides its entertainment value, this book can be used as a supplement in courses in calculus, geometry, and teacher education. Professors of mathematics and physics may find some problems and observations to be useful in their teaching.

Required background. Most of the book (chapters 2–5) requires only precalculus and some basic geometry, and the level of difficulty stays roughly flat throughout those chapters, with a few crests and valleys. Chapters 6 and 7 require only an acquaintance with the derivative and the integral. At the end of chapter 7 I mention the divergence, but in a way that requires no prior exposure. This chapter should be accessible to anyone familiar with precalculus.

The second part (chapters 6–11) uses on rare occasions a few concepts from multivariable calculus, but I tried to avoid the jargon as much as possible, hoping that intuition will help the reader jump over some technical gaps.

Everything one needs from physics is described in the appendix; no prior background is assumed.

This book can be read one section or problem at a time; if you get stuck, it only takes turning a page to gain traction. A few exceptions to this topic-per-page structure occur, mostly in the later chapters.

Sources. Many, but not all *solutions* in this book are, to my knowledge, original. These include solutions to problems 2.6, 2.9, 2.10, 2.11, 2.13, 3.3, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.17, 3.18, 3.19, 3.20, 3.21, 5.2, 5.3, 6.1, 6.2, 6.3, 6.4, 6.5, 7.1, and 7.2. The interpretations in chapter 8 and in sections 9.3, 9.8 and 11.8 appear to be new.

There is not much literature on the topic of this book. When I was in high school, an example from Uspenski's book [U] struck me so much that the topic became a hobby.[4] More problems of the

[4]This is the first example of this book, in section 2.2. Tokieda's article [TO] contains, together with this example, some very nice additional ones.

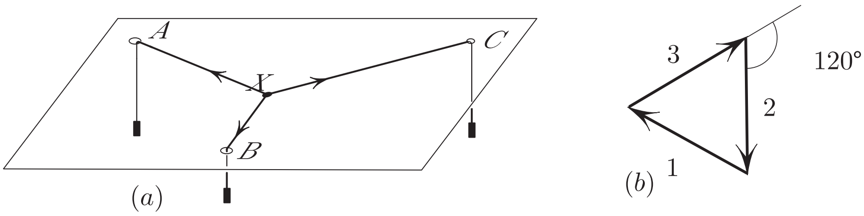

Figure 1.1. If X minimizes total distance $XA + XB + XC$, then the angles at X are 120°.

kind described here are in the small book by Kogan [K] and Balk and Boltyanskii [BB], and in chapter 9 of Polya's book [P]. And the main source of such problems and solutions is the 24-centuries-old work by Archimedes [ARC].

1.3 A Physical versus a Mathematical Solution: An Example

Problem. *Given three points A, B, and C in the plane, find the point X for which the sum of distances $XA + XB + XC$ is minimal.*

Physical approach. We start by drilling three holes at A, B, and C in a tabletop (this is cheaper to do as a thought experiment or at a friend's home). Having tied the three strings together, calling the common point X, I slip each string through a different hole and hang equal weights under the table, as shown in figure 1.1. Let us make each weight equal to 1; the potential energy of the first string is then AX: indeed, to drag X from the hole A to its current position X we have to raise the unit weight by distance AX. We endowed the sum of distances $XA + XB + XC$ with the physical meaning of potential energy. Now, if this length/energy is minimal, then the system is in equilibrium. The three forces of tension acting on X then add up to zero and hence they form a triangle (rather than an open path) if placed head-to-tail, as shown in figure 1.1(b). This triangle is equilateral since the weights are equal, and hence the angle between positive directions of these vectors is 120°. We showed that $\angle AXB = \angle BXC = \angle CXA = 120^\circ$.

Mathematical solution. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{x}$ denote the position vectors of the points A, B, C, and X respectively. We have to minimize the sum of lengths $S(\mathbf{x}) = |\mathbf{x}-\mathbf{a}| + |\mathbf{x}-\mathbf{b}| + |\mathbf{x}-\mathbf{c}|$. To that end, we set partial derivatives of S to zero: $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0$, where $\mathbf{x} = (x, y)$, or, expressing the same condition more compactly and geometrically, we set the gradient $\nabla S = \langle \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y} \rangle = \mathbf{0}$. We now compute ∇S. We have $\frac{\partial}{\partial x}|\mathbf{x}-\mathbf{a}| = \frac{\partial}{\partial x}\sqrt{(x-a_1)^2 + (y-a_2)^2} = (x-a_1)/\sqrt{(x-a_1)^2 + (y-a_2)^2}$, and similarly $\frac{\partial}{\partial y}|\mathbf{x}-\mathbf{a}| = (y - a_2)/\sqrt{(x-a_1)^2 + (y-a_2)^2}$. Thus $\nabla|\mathbf{x}-\mathbf{a}| = (\mathbf{x}-\mathbf{a})/|\mathbf{x}-\mathbf{a}|$ is a unit vector, pointing from A to X. We will denote this vector by $\mathbf{e}_a$. This result came from an explicit calculation, but its physical meaning, borrowed from the physical approach, is simply the force with which X pulls the string. Differentiating the remaining two terms $|\mathbf{x}-\mathbf{b}|$ and $|\mathbf{x}-\mathbf{c}|$ in S we obtain $\nabla S = \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c$, where $\mathbf{e}_b$ and $\mathbf{e}_c$ are defined similarly to $\mathbf{e}_a$. We conclude that the optimal position X corresponds to $\nabla S = \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c = \mathbf{0}$. Thus the unit vectors $\mathbf{e}_a$, $\mathbf{e}_b$, $\mathbf{e}_c$ form an equilateral triangle, and any exterior angle of that triangle, that is, the angle between any pair of our unit vectors, is 120°.

It is fascinating to observe how the difficulty changes shape in passing from one approach to the other. In the mathematical solution, the work goes into a formal manipulation. In the physical approach, the work goes into inventing the right physical model. This pattern is shared by many problems in this book.

Relative advantages of the two approaches.

Physical approach	**Mathematical approach**
Less or no computation	Universal applicability
Answer is often conceptual	Rigor
Can lead to new discoveries	
Less background is required	
Accessible to precalc students	

The physical approach suits some subjects more than others. The subject of complex variables is one example where physical intuition is very fruitful. Some of the fundamental ideas of the subject, such as the Cauchy-Goursat theorem, the Cauchy integral formula, and the Riemann mapping theorem, can be made intuitively obvious in a short time, with minimal physical background. With these ideas Euler's formula

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

acquires a nice interpretation, saying that, for a special incompressible fluid flow in the plane, the fluid injected at the origin at the rate of $\frac{\pi^2}{6}$ gallons per second is absorbed entirely by sinks located at integer points (the details are given in section 11.8 on complex variables). Many such examples can be found in other fields of mathematics, and I hope more will be written on this in the future.

1.4 Acknowledgments

This book would probably not have been written had it not been for something my father said when I was 16. I showed him a physical paradox that had occurred to me, and he said: "Why don't you write it down and start a collection?" This book is an excerpt from this collection, with a few additions.

Many of my friends and colleagues contributed to this book by suggestions and advice. I thank in particular Andrew Belmonte, Alain Chenciner, Charles Conley, Phil Holmes, Vickie Kearn, Nancy Kopell, Paul Nahin, Anna Pierrehumbert, and Sergei Tabachnikov. Thanks to their stimulation the collection was massaged into a presentable form. I am in particular debt to Andy Ruina, who read much of the manuscript and made many suggestions and corrections. I express especial thanks to Tadashi Tokieda, whose extensive suggestions and corrections vastly improved this book.

I gratefully acknowledge support by the National Science Foundation under Grant No. 0605878.

2

THE PYTHAGOREAN THEOREM

2.1 Introduction

Here is a fact seemingly not worth mentioning for its triviality: **Still water in a resting container, with no disturbances, shall remain at rest.** I think it is remarkable that this fact has the Pythagorean theorem as a corollary (p. 17). In addition, this seeming triviality implies the law of sines (p. 18), the Archimedian buoyancy law, and the 3D area version of the Pythagorean theorem (p. 19).

The proof of the Pythagorean theorem, described in section 2.2, suggested a kinematic proof of the Pythagorean theorem, described in section 2.6. The motion-based approach makes some other topics very transparent, including

- The fundamental theorem of calculus.
- The computational formula for the determinant.
- The expansion of the determinant in a row.

All these are described in this chapter.

Several more physical proofs of the Pythagorean theorem are given here, one using springs, and the other using kinetic energy.

The unifying theme of this chapter is the Pythagorean theorem, although we do go off on a few short tangents.

2.2 The "Fish Tank" Proof of the Pythagorean Theorem

Let us build a prism-shaped "fish tank" with our right triangle as the base (figure 2.1). We mount the tank so that it can rotate freely

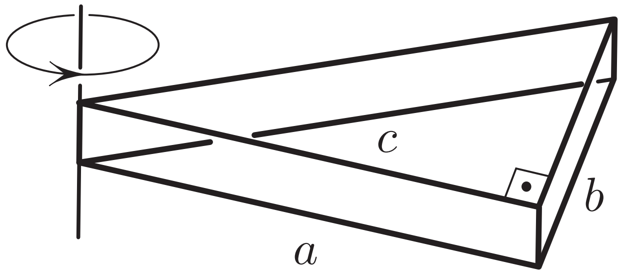

Figure 2.1. The water-filled fish tank, free to rotate around a vertical edge, has no desire to.

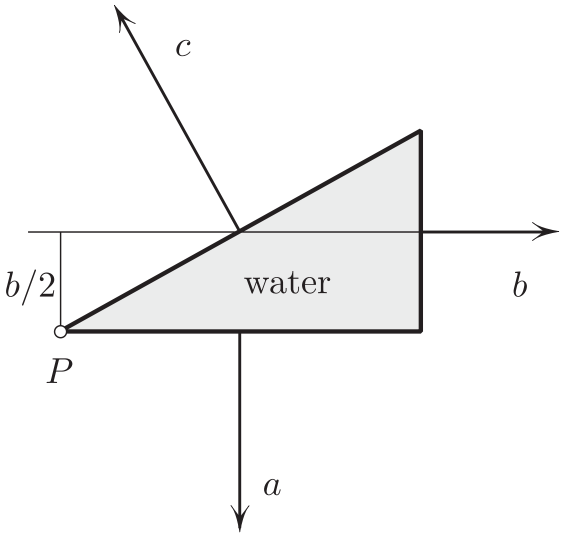

Figure 2.2. The Pythagorean theorem is equivalent to the vanishing of the combined torque upon the tank around *P*. The right-angle property is used here.

around the vertical axis through one end of the hypotenuse. Now let us fill our fish tank with water.

The water pushes on the walls in three competing directions as figure 2.2 shows, each force trying to rotate the tank around P. Of course, the competition is a draw: the tank has zero desire to rotate. Otherwise we would have had an engine which uses no fuel—a so-called perpetual motion machine, forbidden by the law of conservation of energy.

In this case the "desire" is the sum of the three torques of the pressure forces. We note here[1] that the torque of the force around a pivot point P is simply the force's magnitude times the distance from the line of force to the pivot point. The torque measures the intensity with which the force tries to rotate the object it's applied to around P.

For convenience, let us assume the force of pressure to be 1 pound per unit length of the wall—we can always achieve it by adjusting water depth. The three forces are then a, b, and c; the corresponding levers are $a/2$, $b/2$, and $c/2$, and the zero torque condition reads

$$a \cdot \frac{a}{2} + b \cdot \frac{b}{2} - c \cdot \frac{c}{2} = 0, \tag{2.1}$$

or $a^2 + b^2 = c^2$, giving us the Pythagorean theorem!

Still water. Note that we didn't have to build the fish tank, not even in the thought experiment; rather, we can imagine the prism of water embedded in a larger body of water. The Pythagorean theorem follows as before from the fact that the prism will not spontaneously rotate under the pressure of the surrounding fluid on its vertical faces. We conclude that the Pythagorean theorem is a consequence of the fact that still water remains still.

Exercise. *From a point A outside a circle draw a tangent line AT and a secant line APQ as shown in figure 2.3. Prove that*

$$AP \cdot AQ = AT^2. \tag{2.2}$$

Hint: Consider the shaded curvilinear triangle APT in figure 2.3, thought of as a rigid container filled with gas and allowed to pivot around O.

As explained in section 2.3 in a different context, (2.2) expresses the fact that the shaded area remains unchanged under rotations around O. Similarly, the Pythagorean theorem expresses the fact that the area of a right triangle remains unchanged as the triangle is rotated around one of the ends of the hypotenuse.

[1] See section A.5 for full background.

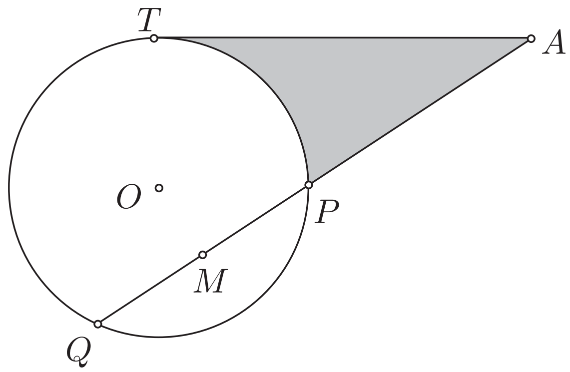

Figure 2.3. Proving $AP \cdot AQ = AT^2$.

2.3 Converting a Physical Argument into a Rigorous Proof

The pivotal[2] point of the "fish tank" proof of the Pythagorean theorem was the vanishing of the net torque around P (figure 2.1). How can we restate this zero-torque idea in purely mathematical terms, without appealing to physical concepts? Here is the answer.

The *physical* statement (2.1) of zero net torque around P translates into the *geometrical* statement that the area of the triangle does not change when the triangle is rotated around P.[3] Here is the proof of this equivalence.

Let $A(\theta)$ be the area of the triangle rotated around P through the angle θ. This area is, of course, independent of θ:

$$A'(\theta) = 0,$$

and we claim that it is this constancy of the area that is equivalent to the zero-torque condition (2.1). To show this equivalence it suffices to show that

$$A'(\theta) = a \cdot \frac{a}{2} + b \cdot \frac{b}{2} - c \cdot \frac{c}{2}. \tag{2.3}$$

[2]This pun was not originally intended.

[3]Here is an example where a trivial-sounding fact (the area of the triangle doesn't change under rotations) hides something less trivial (the Pythagorean theorem.)

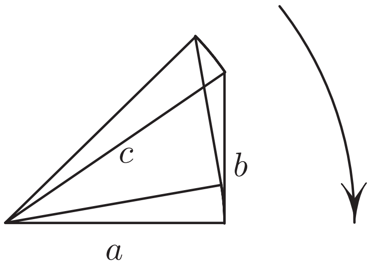

Figure 2.4. The area swept by the two legs equals the area swept by the hypotenuse.

To demonstrate (2.3) we rotate the triangle through a small angle $\Delta\theta$ around P. The side a sweeps a sector of area $\frac{1}{2}a^2\Delta\theta$, with a similar expression for c. In fact, the area swept by b is given by the same expression: $\frac{1}{2}b^2\Delta\theta$. Indeed, b executes two motions simultaneously: (i) sliding in its own direction, contributing nothing to the rate of sweeping of the area, and (ii) rotation around its leading end. We conclude that the area swept is $\frac{1}{2}b^2\Delta\theta$. The total area swept by all three sides is

$$\Delta A = \left(\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}c^2\right)\Delta\theta.$$

Here the minus sign is due to the fact that the area is "lost" through the hypotenuse. Dividing by $\Delta\theta$ and taking the limit as $\Delta\theta \to 0$, we obtain (2.3).

Here are a few other applications of the idea of sweeping:

1. A "ring" proof of the Pythagorean theorem described in section 2.6.
2. A remark on the area between the tracks of two wheels of a bike (section 6.1).
3. A visual proof that the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ equals the area of a parallelogram generated by the vectors $\langle a, c\rangle$ and $\langle b, d\rangle$ (section 2.5).
4. A visual proof of the formula for the row decomposition of a determinant (section 2.5).

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