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Mathematics
for the international student
Mathematics HL (Options)

Including coverage on CD of the
Geometry option for **Further Mathematics SL**

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International Baccalaureate Diploma Programme

MATHEMATICS FOR THE INTERNATIONAL STUDENT
International Baccalaureate Mathematics HL (Options)

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FOREWORD

Mathematics for the International Student: Mathematics HL (Options) has been written as a companion book to the **Mathematics HL (Core)** textbook. Together, they aim to provide students and teachers with appropriate coverage of the two-year Mathematics HL Course (first examinations 2006), which is one of the courses of study in the International Baccalaureate Diploma Programme.

It is not our intention to define the course. Teachers are encouraged to use other resources. We have developed the book independently of the International Baccalaureate Organization (IBO) in consultation with many experienced teachers of IB Mathematics. The text is not endorsed by the IBO.

On the accompanying CD, we offer coverage of the Euclidean Geometry Option for students undertaking the IB Diploma course **Further Mathematics SL**. This Option (with answers) can be printed from the CD.

The interactive features of the CD allow immediate access to our own specially designed geometry packages, graphing packages and more. Teachers are provided with a quick and easy way to demonstrate concepts, and students can discover for themselves and re-visit when necessary.

Instructions appropriate to each graphics calculator problem are on the CD and can be printed for students. These instructions are written for Texas Instruments and Casio calculators.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with associated use of technology, will enhance the students understanding, knowledge and appreciation of mathematics and its universal application.

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PJB PJ PMU
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The publishers wish to make it clear that acknowledging these individuals does not imply any endorsement of this book by any of them, and all responsibility for the content rests with the authors and publishers.

TABLE OF CONTENTS

FURTHER MATHEMATICS SL TOPIC 1

GEOMETRY

Available only by clicking on the icon alongside.
This chapter plus answers is fully printable.

TOPIC 1

HL TOPIC 8

(Further mathematics SL Topic 2)

HL TOPIC 9

(Further mathematics SL Topic 3)

HL TOPIC 10

(Further mathematics SL Topic 4)

HL TOPIC 11

(Further mathematics SL Topic 5)

SYMBOLS AND NOTATION

$\mathrm{E}(X)$	the expected value of X, which is μ
$\mathrm{Var}(X)$	the variance of X, which is σ_X^2
$Z = \dfrac{X-\mu}{\sigma}$	the standardised variable
P(.......)	the probability of .... occurring
$\sim$	is distributed as
$\approx$	is approximately equal to
$\overline{x}$	the sample mean
s_n^2	the sample variance
s_{n-1}^2	the unbiassed estimate of σ^2
μ_X	the mean of random variable X
σ_X	the standard deviation of random variable X
$\mathrm{DU}(n)$	the discrete uniform distribution
$\mathrm{B}(n, p)$	the binomial distribution
$\mathrm{B}(1, p)$	the Bernoulli distribution
$\mathrm{Hyp}(n, M, N)$	the hypergeometric distribution
$\mathrm{Geo}(p)$	the geometric distribution
$\mathrm{NB}(r, p)$	the negative binomial distribution
$\mathrm{Po}(m)$	the Poisson distribution
$\mathrm{U}(a, b)$	the continuous uniform distribution
$\mathrm{Exp}(\lambda)$	the exponential distribution
$\mathrm{N}(\mu, \sigma^2)$	the normal distribution
$\widehat{p}$	the random variable of sample proportions
$\overline{X}$	the random variable of sample means
T	the random variable of the t-distribution
ν	the number of degrees of freedom
H_0	the null hypothesis
H_1	the alternative hypothesis
χ^2_{calc}	the chi-squared statistic

{ .... }	the set of all elements ....
$\in$	is an element of
$\notin$	is not an element of
$\{x \mid$	the set of all x such that ......
$\mathbb{N}$	the set of all natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of all complex numbers
$\mathbb{Z}^+$	the set of positive integers
$\mathbb{P}$	the set of all prime numbers
$\mathbb{U}$	the universal set
$\emptyset$ or $\{\ \}$	the empty (null) set
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$P(A)$	the power of set A
$A \cap B$	the intersection of sets A and B
$A \cup B$	the union of sets A and B
$\Rightarrow$	implies that
$\not\Rightarrow$	does not imply that
A'	the complement of the set A
$n(A)$	the number of elements in the set A
$A \backslash B$	the difference of sets A and B
$A \Delta B$	the symmetric difference of sets A and B
$A \times B$	the Cartesian product of sets A and B
R	a relation of ordered pairs
xRy	x is related to y
$x \equiv y (\mathrm{mod}\, n)$	x is equivalent to y, modulo n
$\mathbb{Z}_n$	the set of residue classes, modulo n
$\times_n$	multiplication, modulo n
$2\mathbb{Z}$	the set of even integers
$f:\ A \to B$	f is a function under which each element of set A has an image in set B
$f:\ x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f

$f \circ g$ or $f(g(x))$	the composite function of f and g
$\lvert x \rvert$	the modulus or absolute value of x
$[\,a\,,\,b\,]$	the closed interval, $a \leqslant x \leqslant b$
$]\,a,\,b\,[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
$\{u_n\}$	the sequence with nth term u_n
S_n	the sum of the first n terms of a sequence
S_∞	the sum to infinity of a series
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + u_3 + ... + u_n$
$\prod_{i=1}^{n} u_i$	$u_1 \times u_2 \times u_3 \times ..... \times u_n$
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\lim_{x \to a+} f(x)$	the limit of $f(x)$ as x tends to a from the positive side of a
$\max\{a,\,b\}$	the maximum value of a or b
$\sum_{n=0}^{\infty} c_n\,x^n$	the power series whose terms have form $c_n\,x^n$
$a \mid b$	a divides b, *or* a is a factor of b
$a \nmid b$	a does not divide b, *or* a is a not a factor of b
$\gcd(a,\,b)$	the greatest common divisor of a and b
$\mathrm{lcm}(a,\,b)$	the least common multiple of a and b
$\cong$	is isomorphic to
$\overline{G}$	is the complement of G
$\mathbf{A}$	matrix $\mathbf{A}$
$\mathbf{A}^n$	matrix $\mathbf{A}$ to the power of n
$\mathbf{A}(G)$	the adjacency matrix of G
$\mathrm{A}(x,\,y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
$\widehat{\mathrm{A}}$	the angle at A
$\widehat{\mathrm{CAB}}$ or $\measuredangle\mathrm{CAB}$	the angle between [CA] and [AB]
$\Delta\mathrm{ABC}$	the triangle whose vertices are A, B and C *or* the area of triangle ABC
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
AB.CD	length AB $\times$ length CD
PT^2	PT $\times$ PT
Power M_C	the power of point M relative to circle C
$\overrightarrow{\mathrm{AB}}$	the vector from A to B

HL Topic 8

(Further Mathematics SL Topic 2)

Before beginning any work on this option, it is recommended that a careful revision of the core requirements for statistics and probability is made.

This is identified by "**Topic 6 – Core: Statistics and Probability**" as expressed in the syllabus guide on pages 26–29 of IBO document on the Diploma Programme Mathematics HL for the first examination 2006.

Throughout this booklet, there will be many references to the core requirements, taken from "Mathematics for the International Student Mathematics HL (Core)" Paul Urban et al, published by Haese and Harris, especially chapters 18, 19, and 30. This will be referred to as "from the text".

Statistics and probability

Contents:

A EXPECTATION ALGEBRA

E(X), THE EXPECTED VALUE OF X

Recall that if a random variable X has mean μ then μ is known as the **expected value** of X, or simply $\mathrm{E}(X)$.

$$\mu = \mathrm{E}(X) = \begin{cases} \sum x P(x), & \text{for discrete } X \\ \int x f(x)\, dx, & \text{for continuous } X \end{cases}$$

From section **30E.1** of the text (*Investigation 1*) we noticed that

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

Proof: (discrete case only)

$$\begin{aligned} \mathrm{E}(aX+b) &= \textstyle\sum (ax+b)P(x) \\ &= \textstyle\sum [axP(x) + bP(x)] \\ &= a\textstyle\sum xP(x) + b\sum P(x) \\ &= a\mathrm{E}(X) + b(1) \qquad \{\text{as } \textstyle\sum P(x) = 1\} \\ &= a\mathrm{E}(X) + b \end{aligned}$$

Var(X), THE VARIANCE OF X

A random variable X, has **variance** σ^2, also known as Var(X)

where $$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}((X-\mu)^2)$$

Notice that for **discrete** X

- $\mathrm{Var}(X) = \sum (x-\mu)^2 p(x)$
- $\mathrm{Var}(X) = \sum x^2 p(x) - \mu^2$
- $\mathrm{Var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$

Again, from *Investigation 1* of **Section 30E.1**, $$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

Proof: (discrete case only)

$$\begin{aligned} \mathrm{Var}(aX+b) &= \mathrm{E}((aX+b)^2) - \{\mathrm{E}(aX+b)\}^2 \\ &= \mathrm{E}\big(a^2X^2 + 2abX + b^2\big) - \{a\mathrm{E}(X)+b\}^2 \\ &= a^2\,\mathrm{E}(X^2) + 2ab\,\mathrm{E}(X) + b^2 - a^2\{\mathrm{E}(X)\}^2 - 2ab\,\mathrm{E}(X) - b^2 \\ &= a^2\mathrm{E}(X^2) - a^2\{\mathrm{E}(X)\}^2 \\ &= a^2[\mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2] \\ &= a^2\mathrm{Var}\,(X) \end{aligned}$$

THE STANDARDISED VARIABLE, Z

If a random variable X is normally distributed with mean μ and variance σ^2 we write $X \sim \mathrm{N}(\mu, \sigma 2)$, where $\sim$ reads *is distributed as*.

The standardised variable Z is defined as $Z = \dfrac{X-\mu}{\sigma}$ and has mean 0 and variance 1.

Proof: The mean of Z is

$$\begin{aligned} &\mathrm{E}(Z) \\ &= \mathrm{E}\left(\tfrac{1}{\sigma}X - \tfrac{\mu}{\sigma}\right) \\ &= \tfrac{1}{\sigma}\mathrm{E}(X) - \tfrac{\mu}{\sigma} \\ &= \tfrac{1}{\sigma}\mu - \tfrac{\mu}{\sigma} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Var}(Z) \\ &= \mathrm{Var}\left(\tfrac{1}{\sigma}X - \tfrac{\mu}{\sigma}\right) \\ &= \left(\tfrac{1}{\sigma}\right)^2 \mathrm{Var}(X) \\ &= \tfrac{1}{\sigma^2} \times \sigma^2 \\ &= 1 \end{aligned}$$

This now gives us a formal basis on which we can standardise a normal variable, as described in the Core text.

Example 1

Suppose the scores in a Mathematics exam are distributed normally with unknown mean μ and standard deviation of 25.5. If only the top 10% of students receive an A, and the cut-off score for an A is any mark greater than 85%, find the mean, μ, of this distribution.

$$\mathrm{P}(X > 85) = 0.1 \qquad \{\text{as } \; 10\% = 0.1\}$$

$$\therefore \; \mathrm{P}(X \leqslant 85) = 0.9$$

$$\therefore \; \mathrm{P}\left(\frac{X-\mu}{25.5} \leqslant \frac{85-\mu}{25.5}\right) = 0.9$$

$$\therefore \; \mathrm{P}\left(Z \leqslant \frac{85-\mu}{25.5}\right) = 0.9$$

$$\therefore \; \frac{85-\mu}{25.5} = \text{invNorm }(0.9)$$

$$\therefore \; \mu = 85 - 25.5 \times \text{invNorm}(0.9)$$

$$\therefore \; \mu \approx 52.3$$

For **two independent random variables** X_1 and X_2 (not necessarily from the same population)

- $\mathbf{E}(a_1X_1 \pm a_2X_2) = a_1\mathbf{E}(X_1) \pm a_2\mathbf{E}(X_2)$
- $\mathbf{Var}(a_1X_1 \pm a_2X_2) = {a_1}^2\mathbf{Var}(X_1) + {a_2}^2\mathbf{Var}(X_2)$

The proof of these results is beyond the scope of this course.

The generalisation of the above is:

For n independent random variables; $X_1, X_2, X_3, X_4, \ldots\ldots X_n$

- $\mathbf{E}(a_1X_1 \pm a_2X_2 \pm \ldots. \pm a_nX_n) = a_1\mathbf{E}(X_1) \pm a_2\mathbf{E}(X_2) \pm \ldots. \pm a_n\mathbf{E}(X_n)$
- $\mathbf{Var}(a_1X_1 \pm a_2X_2 \pm \ldots. \pm a_nX_n) = {a_1}^2\mathbf{Var}(X_1) + {a_2}^2\mathbf{Var}(X_2) + \ldots. + {a_n}^2\mathbf{Var}(X_n)$

Note: These generalised results can be proved using the Principle of Mathematical Induction assuming that the case $n = 2$ is true.

Proof: (by the Principle of Mathematical Induction)

(Firstly for the mean)

(1) When $n = 2$, the result is true (assumed).

(2) If P_k is true, then

$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k) = a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2) \pm \ldots\ldots \pm a_k\mathrm{E}(X_k) \ldots\ldots (*)$

$\therefore \quad \mathrm{E}(a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k \pm a_{k+1}X_{k+1})$

$= \mathrm{E}([a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k] \pm a_{k+1}X_{k+1})$

$= \mathrm{E}([a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k]) \pm \mathrm{E}(a_{k+1}X_{k+1})$ {case $n = 2$}

$= a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2) \pm \ldots\ldots \pm a_k\mathrm{E}(X_k) \pm a_{k+1}\mathrm{E}(X_{k+1})$ {using $(*)$}

Thus P_{k+1} is true whenever P_k is true and $P(2)$ is true.

$\Rightarrow P_n$ is true for all $n \in \mathbb{Z}^+$, $n \geqslant 2$.

(For the variance)

(1) When $n = 2$, the result is true (given).

(2) If P_k is true, then

$\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k)$

$= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots\ldots + a_k^2\mathrm{Var}(X_k) \ldots\ldots (*)$

Now $\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k \pm a_{k+1}X_{k+1})$

$= \mathrm{Var}([a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k] \pm a_{k+1}X_{k+1})$ {case $n = 2$}

$= \mathrm{Var}[a_1X_1 \pm a_2X_2 \pm \ldots\ldots \pm a_kX_k] + \mathrm{Var}(a_{k+1}X_{k+1})$

$= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots\ldots + a_k^2\mathrm{Var}(X_k) + a_{k+1}^2\mathrm{Var}(X_{k+1})$ {using $*$}

Thus P_{k+1} is true whenever P_k is true and P_2 is true.

$\therefore \quad P_n$ is true {Principle of Math. Induction}

Note: Any linear combination of independent normal random variables is itself a normal random variable.

For example, if X_1, X_2 and X_3 are independent normal random variables (RV) then $2X_1 + 3X_2 - 4X_3$ is a normal random variable.

$\mathrm{E}(2X_1 + 3X_2 - 4X_3) = 2\mathrm{E}(X_1) + 3\mathrm{E}(X_2) - 4\mathrm{E}(X_3)$ and

$\mathrm{Var}(2X_1 + 3X_2 - 4X_3) = 4\mathrm{Var}(X_1) + 9\mathrm{Var}(X_2) + 16\mathrm{Var}(X_3)$

Example 2

The weights of male employees in a bank are normally distributed with a mean $\mu = 71.5$ kg and standard deviation $\sigma = 7.3$ kg. The bank has an elevator with a maximum recommended load of 444 kg for safety reasons. Six male employees enter the elevator. Calculate the probability p that their combined weight exceeds the maximum recommended load.

We are concerned with the sum of their weights
and consider $Y = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$ {independent RV's}

Now $\mathrm{E}(Y) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \ldots\ldots + \mathrm{E}(X_6)$

$= 71.5 + 71.5 + \ldots\ldots + 71.5$

$= 6 \times 71.5 = 429$ kg

and $\text{Var}(Y)$
$= \text{Var}(X_1) + \text{Var}(X_2) + ... + \text{Var}(X_6)$
$= 7.3^2 + 7.3^2 + ... + 7.3^2$
$= 6 \times 7.3^2$
$= 319.74$

$\therefore$ Y is normally distributed with mean 429 kg and variance 319.74 kg^2
i.e., $Y \sim \text{N}(429, 319.74)$ $\sigma^2 = 319.74$

Now $\text{P}(Y > 444) = \text{normalcdf}(444, \text{E99}, 429, \sqrt{319.74})$
≈ 0.201

So, there is a 20.1% chance that their combined weight will exceed 444 kg.

Example 3

For **Example 2**, do a suitable calculation to recommend the maximum number of males to use the elevator, given that there should be no more than a 0.1% chance of the total weight exceeding 444 kg.

From **Example 2**, six men is too many as there is a 20.1% chance of overload.
Now we try $n = 5$

$\text{E}(Y)$	$\text{Var}(Y)$
$= 5 \times 71.5$	$= 5 \times 7.3^2$
$= 357.5$ kg	≈ 266.45 kg^2

Now $Y \sim \text{N}(357.5, 266.45)$ i.e., $\sigma^2 = 266.45$

and $\text{P}(Y > 444) = \text{normalcdf}(444, \text{E99}, 357.5, \sqrt{266.45})$
$\approx 5.83 \times 10^{-8}$

So, for $n = 5$ there is much less than a 0.1% chance of the total weight exceeding 444 kg. Hence, we should recommend for safety reasons that a maximum of 5 men use the elevator at the same time.

Example 4

Given three independent samples $X_1 = 2X$, $X_2 = 4 - 3X$, and $X_3 = 4X + 1$, taken from a random distribution X with mean 11 and standard deviation 2, find the mean and standard deviation of the random variable $(X_1 + X_2 + X_3)$.

mean	variance
$= \text{E}(X_1 + X_2 + X_3)$	$= \text{Var}(X_1 + X_2 + X_3)$
$= \text{E}(X_1) + \text{E}(X_2) + \text{E}(X_3)$	$= \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3)$
$= 2\text{E}(X) + 4 - 3\text{E}(X) + 4\text{E}(X) + 1$	$= 4\text{Var}(X) + 9\text{Var}(X) + 16\text{Var}(X)$
$= 3\text{E}(X) + 5$	$= 29\text{Var}(X)$
$= 3(11) + 5$	$= 29 \times 2^2$
$= 38$	$= 116$

$\therefore$ mean is 38 and standard deviation is $\sqrt{116} \approx 10.8$.

Example 5

A cereal manufacturer produces packets of cereal in two sizes, small (S) and economy (E). The amount in each packet is distributed normally and independently as follows:

	Mean (g)	*Variance (g^2)*
Small	315	4
Economy	950	25

a A packet of each size is selected at random. Find the probability that the economy packet contains less than three times the amount of the small packet.

b One economy and three small packets are selected at random.
Find the probability that the amount in the economy packet is less than the total amount in the three small packets.

$S \sim \text{N}(315, 4)$ and $E \sim \text{N}(950, 25)$.

a To find the probability that the economy packet contains less than three times the amount in a small packet we need to calculate $\text{P}(e < 3s)$
i.e., $\text{P}(e - 3s < 0)$

Now $\text{E}(E - 3S)$
$= \text{E}(E) - 3\,\text{E}(S)$
$= 950 - 3 \times (315)$
$= 5$

and $\text{Var}(E - 3S)$
$= \text{Var}(E) + 9\,\text{Var}(S)$
$= 25 + 9 \times 4$
$= 61$

$\therefore\ E - 3S \sim \text{N}(5, 61)$

and $\text{P}(e - 3s < 0) \approx 0.261$ {calculator}

b This time we need to calculate $\text{P}(e < s_1 + s_2 + s_3)$
i.e., $\text{P}(e - (s_1 + s_2 + s_3) < 0)$

Now $\text{E}(E - (S_1 + S_2 + S_3))$
$= \text{E}(E) - 3\,\text{E}(S)$
$= 950 - 3 \times 315$
$= 5$

and $\text{Var}(E - (S_1 + S_2 + S_3))$
$= \text{Var}(E) + \text{Var}(S_1) + \text{Var}(S_2) + \text{Var}(S_3)$
$= 25 + 12$
$= 37$

$\therefore\ E - (S_1 + S_2 + S_3) \sim \text{N}(5, 37)$

and $\text{P}(e - (s_1 + s_2 + s_3)) \approx 0.206$ {calculator}

UNBIASED ESTIMATORS OF MEAN μ AND VARIANCE σ^2 FOR A POPULATION

Often μ and σ for a population are unknown and we may wish to use a representative sample to estimate μ and σ. We observed in section **18F** of the text that:

- $\overline{x}$, the sample mean, gives us an **unbiased** estimate of μ
- $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2$, where s_n^2 is the sample's variance and n is the sample size, gives us an **unbiased** estimate of the population's variance σ^2.

Note: $\overline{x}$ is an **unbiased** estimate of μ if $\mathrm{E}(\overline{X}) = \mu$.

Proof: (that $\overline{x}$ is an unbiased estimate of μ)

$$\begin{aligned}
\mathrm{E}(\overline{X}) &= \mathrm{E}\left(\frac{X_1 + X_2 + X_3 + ... + X_n}{n}\right) \\
&= \mathrm{E}\left(\tfrac{1}{n}(X_1 + X_2 + X_3 + ... + X_n)\right) \\
&= \tfrac{1}{n}\,\mathrm{E}(X_1 + X_2 + X_3 + ... + X_n) \qquad \{\text{assuming independence}\} \\
&= \tfrac{1}{n}\,(\mu + \mu + \mu + ... + \mu) \qquad \{n \text{ of them}\} \\
&= \tfrac{1}{n} \times n\mu \\
&= \mu \qquad \therefore \ \overline{x} \text{ is an unbiased estimate of } \mu.
\end{aligned}$$

Notice also that

$$\begin{aligned}
\mathrm{Var}(\overline{X}) &= \left(\tfrac{1}{n}X_1 + \tfrac{1}{n}X_2 + ... + \tfrac{1}{n}X_n\right) \\
&= \tfrac{1}{n^2}\mathrm{Var}(X_1) + \tfrac{1}{n^2}\mathrm{Var}(X_2) + ... + \tfrac{1}{n^2}\mathrm{Var}(X_n) \\
&= \tfrac{1}{n^2}(\sigma^2 + \sigma^2 + ... + \sigma^2) \qquad \{n \text{ of them}\} \\
&= \tfrac{1}{n^2} \times n\sigma^2
\end{aligned}$$

$$\therefore \quad \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$$

Note: s_{n-1}^2 is an unbiased estimate of σ^2.

To prove this we need to show that $\mathrm{E}(s_{n-1}^2) = \sigma^2$.

Proof:

$$s_n{}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n}\left[\sum_{i=1}^{n} X_i{}^2 - n\overline{X}^2\right] = \frac{1}{n}\sum_{i=1}^{n} X_i{}^2 - \overline{X}^2$$

$$\begin{aligned}
\therefore \ \mathrm{E}(s_n{}^2) &= \frac{1}{n}\,\mathrm{E}\left(\sum_{i=1}^{n} X_i{}^2\right) - \mathrm{E}(\overline{X}^2) \qquad \{\text{assuming independence}\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i{}^2) - \mathrm{E}(\overline{X})^2 \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}(\mathrm{Var}(X_i) + \{\mathrm{E}(X_i)\}^2\right] - \left[\mathrm{Var}(\overline{X}) + \{\mathrm{E}(\overline{X})\}^2\right] \\
&\qquad \{\text{using } \ \mathrm{Var}(Y) = \mathrm{E}(Y^2) - \{\mathrm{E}(Y)\}^2\} \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}(\sigma^2 + \mu^2)\right] - \left[\frac{\sigma^2}{n} + \mu^2\right]
\end{aligned}$$

$$= \frac{1}{n}\left(n\sigma^2 + n\mu^2\right) - \frac{\sigma^2}{n} - \mu^2$$

$$= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$

$$= \sigma^2\left(1 - \frac{1}{n}\right) \quad \text{or} \quad \sigma^2\left(\frac{n-1}{n}\right)$$

But $s_{n-1}^2 = \frac{n}{n-1} s_n^2$ and so $\text{E}\left(s_{n-1}^2\right) = \frac{n}{n-1}\text{E}(s_n{}^2) = \sigma^2$

i.e., s_{n-1}^2 is an unbiased estimate of σ^2.

The following example may be useful for designing a portfolio item.

Example 6

In a gambling game you bet on the outcomes of two spinners. These outcomes are called X and Y and the probability distributions for each spinner are tabled below:

x	-3	-2	3	5
$\text{P}(X = x)$	0.25	0.25	0.25	0.25

y	-3	2	5
$\text{P}(Y = y)$	0.5	0.3	0.2

a Briefly explain why these are *well-defined* probability distributions.

b Find the mean and standard deviation of each random variable.

c Suppose it costs \$1 to get a spinner spun and you receive the dollar value of the outcome. For example, if the result is 3 you win \$3 but if the result is -3 you need to pay an extra \$3. In which game are you likely to achieve a better result? On average, do you expect to win, lose or break even? Use **b** to justify your answer.

d Comment on the differences in standard deviation.

e The players get bored with these two simple games and ask if they can play a \$1 game using the *sum* of the scores obtained on each of the spinners. Complete a table like the one given below to show the probability distribution of $X + Y$. A grid may help you do this.

$X + Y$	-6	-5	....	10
$\text{P}(X + y)$		0.125		

Note: If you score a 10, you receive \$10 after paying out \$1. Effectively you win \$9.

f Calculate the mean and standard deviation of U if $U = X + Y$.

g Are you likely to win, lose or draw in the new game? Use **f** to justify your answer.

a As $\sum P(x) = 1$ in each distribution, each is a well-defined probability distribution.

b $\text{E}(X) = \sum xP(x)$
$= -3(0.25) - 2(0.25) + 3(0.25) + 5(0.25)$
$\therefore \quad \mu_x = 0.75$

$\text{Var}(X) = \text{E}(X^2) - \{\text{E}(X)\}^2$
$= 9(0.25) + 4(0.25) + 9(0.25) + 25(0.25) - 0.75^2$
$= 47 \times 0.25 - 0.75^2$
$= 11.1875$ and so $\sigma_X \approx 3.34$

$\text{E}(Y) = \sum yP(y)$
$= -3(0.5) + 2(0.3) + 5(0.2)$
$\therefore \quad \mu_Y = 0.1$

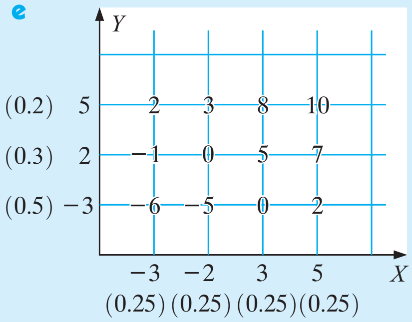
$\text{Var}(Y) = \text{E}(Y^2) - \{\text{E}(Y)\}^2$
$= 9(0.5) + 4(0.3) + 25(0.2) - 0.1^2$
$= 10.69$ and so $\sigma_Y \approx 3.27$

c With X, the expected win is \$0.75 per game. However, it costs \$1 to play so overall there is an expected loss of \$0.25 per game.

With Y, \$0.10 − \$1 = −\$0.90, so there is an expected loss of \$0.90 per game.

d As $\sigma_X > \sigma_Y$ we expect a greater variation in the results of game X.

e



$P(-6) = 0.25 \times 0.5 = 0.125$
$P(-5) = 0.25 \times 0.5 = 0.125$
$P(-1) = 0.25 \times 0.3 = 0.075$
$P(0) = 0.25 \times 0.5 + 0.25 \times 0.3 = 0.200$
$P(2) = 0.25 \times 0.5 + 0.25 \times 0.2 = 0.175$
$P(3) = 0.25 \times 0.2 = 0.050$
$P(5) = 0.25 \times 0.3 = 0.075$
$P(7) = 0.25 \times 0.3 = 0.075$
$P(8) = 0.25 \times 0.2 = 0.050$
$P(10) = 0.25 \times 0.2 = 0.050$

$X+Y$	−6	−5	−1	0	2	3	5	7	8	10	$\sum$
$P(X+Y)$	0.125	0.125	0.075	0.200	0.175	0.050	0.075	0.075	0.050	0.050	1.000

f If $U = X + Y$
$\text{E}(U) = -6(0.125) - 5(0.125) - 1(0.075) + 0 + 2(0.175) + 3(0.050) + 5(0.075) + 7(0.075) + 8(0.050) + 10(0.050)$
$\therefore \quad \mu_U = 0.85$

$\text{Var}(U) = 36(0.125) + 25(0.125) + 1(0.075) + 4(0.175) + 9(0.050) + 25(0.075) + 49(0.075) + 64(0.050) + 100(0.050) - (0.85)^2$
$= 21.8775$
$\therefore \quad \sigma_U = \sqrt{21.8775} \approx 4.68$

g With the new game the expected loss is \$0.15 per game. {\$0.85 − \$1}

EXERCISE 8A

1 Given two independent random variables X and Y whose means and standard deviations are given in the table:

	mean	*s.d.*
X	3.8	0.323
Y	5.7	1.02

a find the mean and standard deviation of $3X - 2Y$

b find the $\text{P}(3X - 2Y > 3)$, given that X and Y are distributed normally. You need to know that any linear combination of independent normal random variables is also normal.

2 X and Y are independent normal random variables with $X \sim \text{N}(-10, 1)$ and $Y \sim (25, 25)$. Find:

a the mean and standard deviation of the random variable $U = 3X + 2Y$.

b $\text{P}(U < 0)$.

3 The marks in an IB Mathematics HL exam are distributed normally with mean μ and standard deviation σ. If the cut off score for a 7 is a mark of 80%, and 10% of students get a 7, and the cut off score for a 6 is a mark of 65% and 30% of students get a 6 or 7, find the mean and standard deviation of the marks in this exam.

4 In a lift, the maximum recommended load is 440 kg. The weights of men are distributed normally with mean 61 kg and standard deviation of 11 kg. The weights of children are also normally distributed with mean 48 kg and standard deviation of 4 kg.

Find the probability that the lift containing 4 men and 3 children will be unsafe. What assumption have you made in your calculation?

5 A coffee machine dispenses white coffee made up of black coffee distributed normally with mean 120 mL and standard deviation 7 mL, and milk distributed normally with mean 28 mL and standard deviation 4.5 mL.

Each cup is marked to a level of 135.5 mL, and if this is not attained then the customer will receive a cup of white coffee free of charge.

Determine whether or not the proprietor should adjust the settings on her machine if she wishes to give away no more than 1% in "free coffees".

6 A drinks manufacturer independently produces bottles of drink in two sizes, small (S) and large (L). The amount in each bottle is distributed normally as follows:

$S \sim \text{N}(280 \text{ mL}, 4 \text{ mL}^2)$ and $L \sim \text{N}(575 \text{ mL}, 16 \text{ mL}^2)$

a When a bottle of each size is selected at random, find the probability that the large bottle contains less than two times the amount in the small bottle.

b One large and two small bottles are selected at random. Find the probability that the amount in the large bottle is less than the total amount in the two small bottles.

7 Chocolate bars are produced independently in two sizes, small (S) and large (L). The amount in each bar is distributed normally as follows:

$S \sim \text{N}(21, 5)$ and $L \sim \text{N}(90, 15)$

a One of each type of bar is selected at random. Find the probability that the large bar contains more than five times the amount in the small bar.

b One large and five small bars are selected at random. Find the probability that the amount in the large bar is more than the total amount in the five small bars.

B CUMULATIVE DISTRIBUTION FUNCTIONS

We will examine **cumulative distribution functions (cdf)** for both **discrete random variables (drv)** and **continuous random variables (crv).**

Definition: The **cumulative distribution function (cdf)** of a random variable X is the probability that X takes a value less than or equal to x,
i.e., $F(x) = \text{P}(X \leqslant x)$.

Recall that a random variable is
- **discrete** if you can *count* the outcomes
- **continuous** if you can *measure* the outcomes.

Example 7

Classify the following as a discrete or continuous random variable:

a the outcomes when you roll an unbiased die
b the heights of students studying the final year of high school
c the outcomes from the two spinners in **Example 6**.

a discrete as you can count them
b continuous as you measure them
c discrete as you can count them

DISCRETE RANDOM VARIABLES

A discrete random variable X has a **probability mass function** given by $p_x = \text{P}(X = x)$ where x is one of the possible outcomes.

A probability mass function of a discrete random variable must be **well-defined**,

$$\text{i.e.,} \quad \sum_{i=1}^{n} p_i = 1 \quad \text{and} \quad 0 \leqslant p_i \leqslant 1 \quad \text{for} \quad i = 1, 2, 3, \ldots.., n.$$

The **cumulative distribution function (cdf)** of a **discrete** random variable X is the probability that X takes a value less than or equal to x,

$$\text{i.e.,} \quad F(x) = \text{P}(X \leqslant x) = \sum_{y \leqslant x} \text{P}(X = y)$$

For example, consider

- tossing one coin, where X is the number of 'heads' resulting
  $X = 0$ or 1 and $F(0) = \text{P}(X \leqslant 0) = \text{P}(X = 0) = \frac{1}{2}$
  $F(1) = \text{P}(X \leqslant 1) = \text{P}(X = 0 \text{ or } 1) = 1$
- tossing two coins, where X is the number of 'heads' resulting
  $X = 0, 1$ or 2 and $F(0) = \text{P}(X \leqslant 0) = \text{P}(X = 0) = \frac{1}{4}$
  $F(1) = \text{P}(X \leqslant 1) = \text{P}(X = 0 \text{ or } 1) = \frac{3}{4}$
  $F(2) = \text{P}(X \leqslant 2) = \text{P}(X = 0, 1 \text{ or } 2) = 1$

TYPES OF DISCRETE RANDOM VARIABLES

DISCRETE UNIFORM

For a **discrete uniform** random variable, the probability mass function takes the same value for all outcomes x.

For example, when rolling a fair (unbiased) die the sample space is $\{1, 2, 3, 4, 5, 6\}$ and $p_x = \frac{1}{6}$ for all x.

The name 'uniform' comes from the fact that p_x values do not change as x changes.

If we are interested in getting a result smaller than 5, we are concerned with the cdf and in this case $$P(X < 5) = P(X \leqslant 4) = F(4) = 4 \times \tfrac{1}{6} = \tfrac{2}{3}$$

If X is a discrete uniform random variable with n distinct outcomes, 1, 2, 3, 4, ..., n, we write $X \sim \text{DU}(n)$.

Note: The outcomes do not have to be 1, 2, 3, 4, ..., n.

This is illustrated in **Example 6** where the random variable X had four possible outcomes -3, -2, 3 and 5.

BINOMIAL

The binomial distribution was observed in **Section 30F** of the Core HL text.

For the **binomial distribution**, the probability mass function is

$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ where n is the number of independent trials,
x is the number of successes in n trials,
p is the probability of success in one trial.

The cdf is $F(x) = P(X \leqslant x) = \sum_{r=0}^{x} \binom{n}{r} p^r (1-p)^{n-r}$.

We write $X \sim B(n, p)$ to indicate that X is distributed binomially. Note that a binomial distribution occurs in *sampling with replacement.*

BERNOULLI

A **Bernoulli distribution** is a binomial distribution where only one trial is conducted, i.e., $n = 1$.

$$P(X = x) = p^x (1-p)^{1-x}, \quad \text{where} \quad x = 0 \text{ or } 1$$

The cdf is $F(x) = P(X \leqslant x) = \sum_{r=0}^{x} p^r (1-p)^{1-r}$, where $x = 0$ or 1.

Hence, a binomial distribution consists of n independent Bernoulli trials.

Note: If $x = 0$, $F(0) = P(x \leqslant 0) = p^0 (1-p)^1 = 1 - p$

If $x = 1$, $F(1) = P(x \leqslant 1) = P(X = 0 \text{ or } X = 1) = 1 - p + p^1(1-p)^0$
$= 1 - p + p$
$= 1$

Discuss what this means.

We write $X \sim B(1, p)$ to indicate that X is Bernoulli distributed.

EXERCISE 8B.1

Uniform, Binomial, Bernoulli Distribution Refer to Core Text **Exercise 19H,** pages 515-516.

1 The discrete random variable X is such that $\text{P}(X = x) = k$, for $X = 5, 10, 15, 20, 25, 30$. Find:

a the probability distribution of x

b μ, the expected value of X

c $\text{P}(X < \mu)$

d σ, the standard deviation of X.

2 Given the random variable X such that $X \sim \text{B}(7, p)$ and $\text{P}(X = 4) = 0.097\,24$, find $P(X = 2)$ where $p < 0.5$.

3 In parts of the USA the probability that it will rain on any given day in August is 0.35. Calculate the probability that in a given week in August in that part of the USA, it will rain on:

a exactly 3 days

b at least 3 days

c at most 3 days

d exactly 3 days in succession.

State any assumptions made in your calculations.

4 A box contains a very large number of red and blue pens. The probability that a pen is blue is 0.8. How many pens would you need to select to be more than 90% certain of picking at least one red pen? State any assumptions made in your calculations.

5 A satellite relies on solar cells for its operation and will be powered provided at least one of its cells is working. Solar cells operate independently of each other, and the probability that an individual cell operates within one year is 0.3.

a For a satellite with 15 solar cells, find the probability that all 15 cells fail within one year.

b For a satellite with 15 solar cells, find the probability that the satellite is still operating at the end of one year.

c For the satellite with n solar cells, find the probability that it is still operating at the end of one year. Hence, find the smallest number of cells required so that the probability of the satellite still operating at the end of one year is at least 0.98.

6 Seventy percent (70%) of the mail to ETECH Couriers is addressed to the Accounts Department.

a In a batch of 20 letters, what is the probability that there will be at least 11 letters to the Accounts Department?

b On average 70 letters arrive each day. What is the mean and standard deviation of the number of letters to the Accounts Department?

7 The table shown gives information about the destination and type of parcels handled by ETECH Couriers.

Destination		*Priority*	*Standard*
Local	40%	70%	30%
Country	20%	45%	55%
Interstate	25%	70%	30%
International	15%	40%	60%

a What is the probability that a parcel is being sent interstate given that it is priority paid?

(**Hint:** Use **Bayes theorem**: refer HL Core text, page 528)

b If two standard parcels are selected, what is the probability that only one will be leaving the state (i.e., Interstate or International)?

Note: The table on **page 31** can be used in the following question.

8 At a school fete fundraiser, an unbiased spinning wheel has numbers 1 to 50 inclusive.

a What is the mean expected score obtained on this wheel during the day?

b What is the standard deviation of the scores obtained during the day?

c What is the probability of getting a multiple of 7 in one spin of the wheel?

If the wheel is spun 500 times during the day:

d What is the likelihood of getting a multiple of 7 more than 15% of the time?

Given that 20 people play each time the wheel is spun, and when a multiple of 7 comes up \$5 is paid to players, but when it does not the players must pay \$1:

e How much would the wheel be expected to make or lose for the school if it was spun 500 times?

f What are the chances the school would lose if the wheel was spun 500 times?

HYPERGEOMETRIC

If we are *sampling without replacement* then we have a **hypergeometric distribution**.

Finding the probability mass function involves the use of combinations to count possible outcomes. Probability questions of this nature were in the Core HL text.

Example 8

A class of IB students contains 10 females and 9 males. A student committee of three is to be randomly chosen. If X is the number of females on the committee, find: **a** $\text{P}(X = 0)$ **b** $\text{P}(X = 1)$ **c** $\text{P}(X = 2)$ **d** $\text{P}(X = 3)$

The total number of unrestricted committees $= \binom{19}{3}$ or C_3^{19}

{as there are 19 students to choose from and we want any 3 of them}

a The number of committees consisting of 0 females and 3 males is $\binom{10}{0}\binom{9}{3}$ $\quad \therefore \text{P}(X = 0) = \dfrac{\binom{10}{0}\binom{9}{3}}{\binom{19}{3}}$

b Likewise, $\text{P}(X = 1) = \dfrac{\binom{10}{1}\binom{9}{2}}{\binom{19}{3}}$

c $\text{P}(X = 2) = \dfrac{\binom{10}{2}\binom{9}{1}}{\binom{19}{3}}$ **d** $\text{P}(X = 3) = \dfrac{\binom{10}{3}\binom{9}{0}}{\binom{19}{3}}$

From **Example 8**, notice that we can write all four possible results in the form

$$\text{P}(X = x) = \frac{\binom{10}{x}\binom{9}{3-x}}{\binom{19}{3}} \quad \text{where} \quad x = 0, 1, 2 \text{ or } 3.$$

This is the probability mass function for this example.

In general:

If we have a population of size N consisting of two types with size M and $N - M$ respectively, and we take a sample of size n *without replacement*, then for the random variable X consisting of how many of M we want to include in the sample, the **hypergeometric distribution** has probability mass function

$$\mathrm{P}(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad \text{where} \quad x=0,\,1,\,2,\,3,\,\ldots\ldots,\,\text{Min}\,(n,\,M)$$

The cdf is $F(x)=\mathrm{P}(X\leqslant x)=\sum_{r=0}^{x}\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ for $x\leqslant n,\,M.$

We write $X\sim\text{Hyp}(n,\,M,\,N)$ to show that X is hypergeometrically distributed.

GEOMETRIC

Consider the following:

A sports magazine gives away photographs of famous football players. 15 photographs are randomly placed in every 100 magazines.

Consider X, the number of magazines you purchase before you get a photograph.

$\mathrm{P}(X=1)=$ P(the first magazine contains a photo) $=0.15$
$\mathrm{P}(X=2)=$ P(the second magazine contains a photo) $=0.85\times0.15$
$\mathrm{P}(X=3)=$ P(the third magazine contains a photo) $=(0.85)^2\times0.15$

So, $\mathrm{P}(X=4)=(0.85)^3\times0.15$, $\mathrm{P}(X=5)=(0.85)^4\times0.15$, etc.

This is an example of a geometric distribution.

If X is the number of trials needed to get a successful outcome, then X is a **geometric discrete** random variable and has probability mass function

$\mathrm{P}(X=x)=p(1-p)^{x-1}$ where $x=1,\,2,\,3,\,4,\,\ldots\ldots$

The cdf is $F(x)=\mathrm{P}(X\leqslant x)=\sum_{r=1}^{x}p(1-p)^{r-1}$ for $r=1,\,2,\,3,\,4,\,\ldots\ldots$

We write $X\sim\text{Geo}(p)$ to show that X is a geometric discrete random variable.

Example 9

In a spinning wheel game with numbers 1 to 50 on the wheel, you win if you get a multiple of 7. Assuming the game is fair, find the probability that you win:

a after exactly four games
b if you need at most four games
c after no more than three games
d after more than three games.

If X is the number of games played until you win
then $X\sim\text{Geo}(p)$ where $p=\frac{7}{50}=0.14$ and $1-p=0.86$

a $\mathrm{P}(X=4)$
$=p(1-p)^3$
$=0.14\times(0.86)^3$
≈0.0890

b P(need at most four games)
$=\mathrm{P}(X\leqslant4)$
$=p+p(1-p)+p(1-p)^2+p(1-p)^3$
$=p\left[1+(1-p)+(1-p)^2+(1-p)^3\right]$
$=0.14\left[1+0.86+0.86^2+0.86^3\right]$
≈0.453

Note: $\text{P}(X \leqslant 4) = \text{P(win in one of the first four games)}$
$= 1 - \text{P(does not win in first four games)}$
$= 1 - (1-p)^4$
$= 1 - (0.86)^4$ which ≈ 0.453

gives us an alternative method of calculation.

c P(wins after no more than three games)
$= \text{P}(X \leqslant 3)$
$= 1 - \text{P(does not win in one of the first three games)}$
$= 1 - (1-p)^3$
$= 1 - 0.86^3$
≈ 0.364

d P(wins after more than 3 games) $= \text{P}(X > 3)$
$= 1 - \text{P}(X \leqslant 3)$
$\approx 1 - 0.364$ {from **c**}
≈ 0.636

Note:
- In **Example 9** we observed that if $X \sim \text{Geo}(p)$ then

$$\text{P}(X \leqslant x) = \sum_{r=1}^{x} p(1-p)^{r-1} = 1 - (1-p)^x.$$

Can you prove this result algebraically?

Hint: $\text{P}(X \leqslant x) = \sum_{r=1}^{x} p(1-p)^{r-1} = p\sum_{r=1}^{x}(1-p)^{r-1}$

and $\sum_{r=1}^{x}(1-p)^{r-1}$ is a geometric series.

- The **modal score** (the score with the highest probability of occurring) for a geometric random variable is always $x = 1$. Can you explain why?

Example 10

Show that if $X \sim \text{Geo}(p)$ then $\sum_{i=1}^{\infty} \text{P}(X = i) = 1$.

$$\begin{aligned}
\sum_{i=1}^{\infty} \text{P}(X = i) &= \text{P}(X=1) + \text{P}(X=2) + \text{P}(X=3) + \ldots\ldots \\
&= p(1-p)^0 + p(1-p)^1 + p(1-p)^2 + \ldots\ldots \\
&= p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \ldots\ldots\right] \\
&= p\left(\frac{1}{1-(1-p)}\right) \quad \text{as we have an infinite GS with } u_1 = 1 \text{ and } r = 1-p \text{ where } 0 < r < 1 \\
&= p\left(\frac{1}{p}\right) \\
&= 1
\end{aligned}$$

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