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# **Challenging Problems in Geometry**

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Preparing to Solve a Problem

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The challenge of well-posed problems transcends national boundaries, ethnic origins, political systems, economic doctrines, and religious beliefs; the appeal is almost universal. Why? You are invited to formulate your own explanation. We simply accept the observation and exploit it here for entertainment and enrichment.

This book is a new, combined edition of two volumes first published in 1970. It contains nearly two hundred problems, many with extensions or variations that we call *challenges*. Supplied with pencil and paper and fortified with a diligent attitude, you can make this material the starting point for exploring unfamiliar or little-known aspects of mathematics. The challenges will spur you on; perhaps you can even supply your own challenges in some cases. A study of these nonroutine problems can provide valuable underpinnings for work in more advanced mathematics.

This book, with slight modifications made, is as appropriate now as it was a quarter century ago when it was first published. The National Council of Teachers of Mathematics (NCTM), in the *Curriculum and Evaluation Standards for High School Mathematics* (1989), lists problem solving as its first standard, stating that “mathematical problem solving in its broadest sense is nearly synonymous with doing mathematics.” They go on to say, “[problem solving] is a process by which the fabric of mathematics is identified in later standards as both constructive and reinforced.”

This strong emphasis on mathematics is by no means a new agenda item. In 1980, the NCTM published *An Agenda for Action*. There, the NCTM also had problem solving as its first item, stating “educators should give priority to the identification and analysis of specific problem solving strategies.... [and] should develop and disseminate examples of ‘good problems’ and strategies.” It is our intention to provide secondary mathematics educators with materials to help them implement this very important recommendation.

## ABOUT THE BOOK

*Challenging Problems in Geometry* is organized into three main parts: “Problems,” “Solutions,” and “Hints.” Unlike many contemporary problem-solving resources, this book is arranged *not* by problem-solving technique, but by topic. We feel that announcing the technique to be used stifles creativity and destroys a good part of the fun of problem solving.

The problems themselves are grouped into two sections. [Section I](#), “A New Twist on Familiar Topics,” covers five topics that roughly parallel the sequence of the high school geometry course. [Section II](#), “Further Investigations,” presents topics not generally covered in the high school geometry course, but certainly within the scope of that audience. These topics lead to some very interesting extensions and enable the reader to investigate numerous fascinating geometric relationships.

Within each topic, the problems are arranged in approximate order of difficulty. For some problems, the basic difficulty may lie in making the distinction between relevant and irrelevant data or between known and unknown information. The sure ability to make these distinctions is part of the process of problem solving, and each devotee must develop this power by him- or herself. It will come with sustained effort.

In the “Solutions” part of the book, each problem is restated and then its solution is given. Answers are also provided for many but not all of the challenges. In the solutions (and later in the hints), you will notice citations such as “(#23)” and “(Formula #5b).” These refer to the definition

postulates, and theorems listed in [Appendix I](#), and the formulas given in [Appendix II](#).

~~From time to time we give alternate methods of solution, for there is rarely only one way to solve a problem. The solutions shown are far from exhaustive, and intentionally so, allowing you to try a variety of different approaches. Particularly enlightening is the strategy of using multiple methods integrating algebra, geometry, and trigonometry. Instances of multiple methods or multiple interpretations appear in the solutions. Our continuing challenge to you, the reader, is to find a different method of solution for every problem.~~

The third part of the book, "[Hints](#)," offers suggestions for each problem and for selected challenges. Without giving away the solution, these hints can help you get back on the track if you run into difficulty.

## **USING THE BOOK**

This book may be used in a variety of ways. It is a valuable supplement to the basic geometry textbook, both for further explorations on specific topics and for practice in developing problem solving techniques. The book also has a natural place in preparing individuals or student teams for participation in mathematics contests. Mathematics clubs might use this book as a source for independent projects or activities. Whatever the use, experience has shown that these problems motivate people of all ages to pursue more vigorously the study of mathematics.

Very near the completion of the first phase of this project, the passing of Professor Charles T. Salkind grieved the many who knew and respected him. He dedicated much of his life to the study of problem posing and problem solving and to projects aimed at making problem solving meaningful, interesting, and instructive to mathematics students at all levels. His efforts were praised by all. Working closely with this truly great man was a fascinating and pleasurable experience.

Alfred S. Posamenti

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# PREPARING TO SOLVE A PROBLEM

A strategy for attacking a problem is frequently dictated by the use of analogy. In fact, searching for an analogue appears to be a psychological necessity. However, some analogues are more apparent than real, so analogies should be scrutinized with care. Allied to analogy is structural similarity or pattern. Identifying a pattern in apparently unrelated problems is not a common achievement, but when done successfully it brings immense satisfaction.

Failure to solve a problem is sometimes the result of fixed habits of thought, that is, inflexible approaches. When familiar approaches prove fruitless, be prepared to alter the line of attack. A flexible attitude may help you to avoid needless frustration.

Here are three ways to make a problem yield dividends:

- (1) The result of formal manipulation, that is, “the answer,” may or may not be meaningful; find it out! Investigate the possibility that the answer is not unique. If more than one answer is obtained, decide on the acceptability of each alternative. Where appropriate, estimate the answer in advance of the solution. The habit of estimating in advance should help to prevent crude errors in manipulation.
- (2) Check possible restrictions on the data and/or the results. Vary the data in significant ways and study the effect of such variations on the original result.
- (3) The insight needed to solve a generalized problem is sometimes gained by first specializing. Conversely, a specialized problem, difficult when tackled directly, sometimes yields to an easy solution by first generalizing it.

As is often true, there may be more than one way to solve a problem. There is usually what we will refer to as the “peasant’s way” in contrast to the “poet’s way”—the latter being the more elegant method.

To better understand this distinction, let us consider the following problem:

If the sum of two numbers is 2, and the product of these same two numbers is 3, find the sum of the reciprocals of these two numbers.

Those attempting to solve the following pair of equations simultaneously are embarking on the “peasant’s way” to solve this problem.

$$\begin{aligned}x + y &= 2 \\xy &= 3\end{aligned}$$

Substituting for  $y$  in the second equation yields the quadratic equation,  $x^2 - 2x + 3 = 0$ . Using the quadratic formula we can find  $x = 1 \pm i\sqrt{2}$ . By adding the reciprocals of these two values of  $x$ , the answer  $\frac{2}{3}$  appears. This is clearly a rather laborious procedure, not particularly elegant.

The “poet’s way” involves working backwards. By considering the desired result

$$\frac{1}{x} + \frac{1}{y}$$

and seeking an expression from which this sum may be derived, one should inspect the algebraic sum

$$\frac{x+y}{xy}$$

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The answer to the original problem is now obvious! That is, since  $x + y = 2$  and  $xy = 3$ ,  $\frac{x+y}{xy} = \frac{2}{3}$ . This is clearly a more elegant solution than the first one.

The “poet’s way” solution to this problem points out a very useful and all too often neglected method of solution. A reverse strategy is certainly not new. It was considered by Pappus of Alexandria about 320 A.D. In Book VII of Pappus’ *Collection* there is a rather complete description of the methods of “analysis” and “synthesis.” T. L. Heath, in his book *A Manual of Greek Mathematics* (Oxford University Press, 1931, pp. 452-53), provides a translation of Pappus’ definitions of these terms:

*Analysis* takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were already done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until, by so retracing our steps, we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backward.

But in *synthesis*, reversing the progress, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of that which was sought: and this we call *synthesis*.

Unfortunately, this method has not received its due emphasis in the mathematics classroom. We hope that the strategy recalled here will serve you well in solving some of the problems presented in this book.

Naturally, there are numerous other clever problem-solving strategies to pick from. In recent years a plethora of books describing various problem-solving methods have become available. A concise description of these problem-solving strategies can be found in *Teaching Secondary School Mathematics: Techniques and Enrichment Units*, by A. S. Posamentier and J. Stepelman, 4th edition (Columbus, Ohio: Prentice Hall/Merrill, 1995).

Our aim in this book is to strengthen the reader’s problem-solving skills through nonroutine motivational examples. We therefore allow the reader the fun of finding the best path to a problem solution, an achievement generating the most pleasure in mathematics.

### A New Twist on Familiar Topics

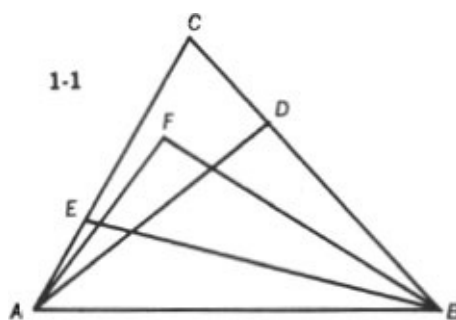
#### 1. Congruence and Parallelism

The problems in this section present applications of several topics that are encountered early in the formal development of plane Euclidean geometry. The major topics are congruence of line segments, angles, and triangles and parallelism in triangles and various types of quadrilaterals.

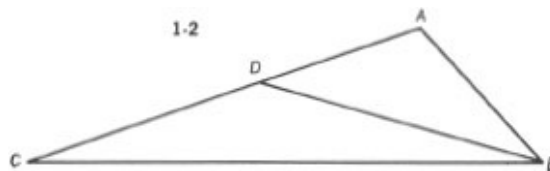
- 1-1** In any  $\triangle ABC$ ,  $E$  and  $D$  are interior points of  $\overline{AC}$  and  $\overline{BC}$ , respectively (Fig. 1-1).  $\overline{AF}$  bisects  $\angle CAB$  and  $\overline{BF}$  bisects  $\angle CBE$ . Prove  $m\angle AEB + m\angle ADB = 2m\angle AFB$ .

**Challenge 1** Prove that this result holds if  $E$  coincides with  $C$ .

**Challenge 2** Prove that the result holds if  $E$  and  $D$  are exterior points on extensions of  $\overline{AC}$  and  $\overline{BC}$  through  $C$ .



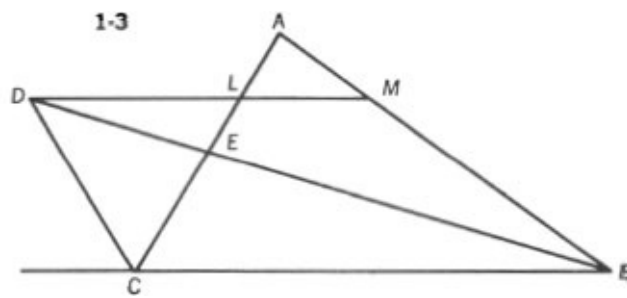
- 1-2** In  $\triangle ABC$ , a point  $D$  is on  $\overline{AC}$  so that  $AB = AD$  (Fig. 1-2).  $m\angle ABC - m\angle ACB = 30$ . Find  $m\angle CBD$ .



- 1-3** The interior bisector of  $\angle B$ , and the exterior bisector of  $\angle C$  of  $\triangle ABC$  meet at  $D$  (Fig. 1-3). Through  $D$ , a line parallel to  $\overline{CB}$  meets  $\overline{AC}$  at  $L$  and  $\overline{AB}$  at  $M$ . If the measures of legs  $\overline{LC}$  and  $\overline{MB}$  of trapezoid  $CLMB$  are 5 and 7, respectively, find the measure of base  $\overline{LM}$ . Prove your result.

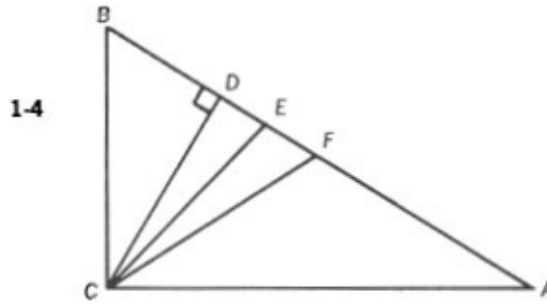
**Challenge** Find  $\overline{LM}$  if  $\triangle ABC$  is equilateral.





1-4 In right  $\triangle ABC$ ,  $\overline{CF}$  is the median to hypotenuse  $\overline{AB}$ ,  $\overline{CE}$  is the bisector of  $\angle ACB$ , and  $\overline{CD}$  is the altitude to  $\overline{AB}$  (Fig. 1-4). Prove that  $\angle DCE \cong \angle ECF$ .

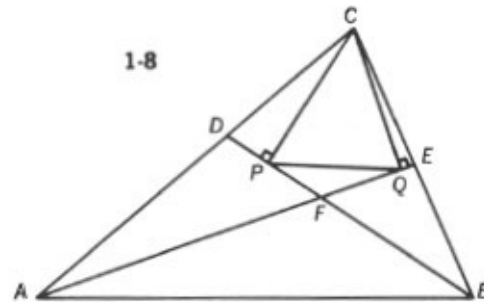
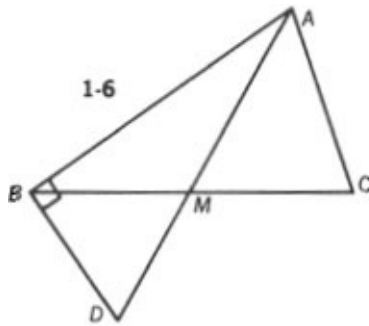
**Challenge** Does this result hold for a non-right triangle?



1-5 The measure of a line segment  $\overline{PC}$ , perpendicular to hypotenuse  $\overline{AC}$  of right  $\triangle ABC$ , is equal to the measure of leg  $BC$ . Show  $BP$  may be perpendicular or parallel to the bisector of  $\angle A$ .

1-6 Prove the following: if, in  $\triangle ABC$ , median  $\overline{AM}$  is such that  $m\angle BAC$  is divided in the ratio 1:1 and  $\overline{AM}$  is extended through  $M$  to  $D$  so that  $\angle DBA$  is a right angle, then  $AC = \frac{1}{2} AD$  (Fig. 1-6).

**Challenge** Find two ways of proving the theorem when  $m\angle A = 90^\circ$ .



1-7 In square  $ABCD$ ,  $M$  is the midpoint of  $\overline{AB}$ . A line perpendicular to  $\overline{MC}$  at  $M$  meets  $\overline{AD}$  at  $K$ . Prove that  $\angle BCM \cong \angle KCM$ .

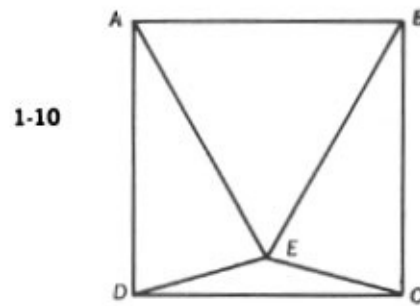
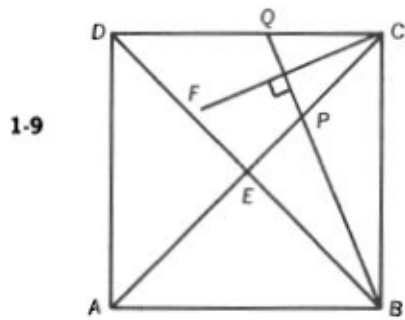
**Challenge** Prove that  $\triangle KDC$  is a 3-4-5 right triangle.

1-8 Given any  $\triangle ABC$ ,  $\overline{AE}$  bisects  $\angle BAC$ ,  $\overline{BD}$  bisects  $\angle ABC$ ,  $\overline{CP} \perp \overline{BD}$ , and  $\overline{CQ} \perp \overline{AE}$  (Fig. 1-8), prove that  $\overline{PQ}$  is parallel to  $\overline{AB}$ .

**Challenge** Identify the points  $P$  and  $Q$  when  $\triangle ABC$  is equilateral.

1-9 Given that  $ABCD$  is a square,  $\overline{CF}$  bisects  $\angle ACD$ , and  $\overline{BPQ}$  is perpendicular to  $CF$  (Fig. 1-9).

prove  $DQ = 2PE$ .

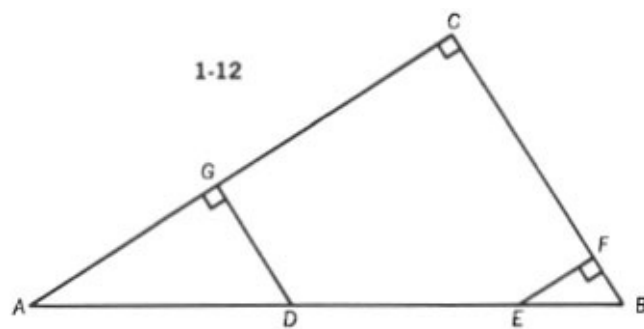
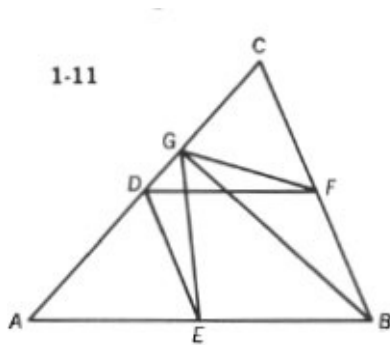


**1-10** Given square  $ABCD$  with  $m\angle EDC = m\angle ECD = 15$ , prove  $\triangle ABC$  is equilateral (Fig. 1-10).

**1-11** In any  $\triangle ABC$ ,  $D$ ,  $E$ , and  $F$  are midpoints of the sides  $\overline{AC}$ ,  $\overline{AB}$ , and  $\overline{BC}$ , respectively (Fig. 1-11).  $\overline{BG}$  is an altitude of  $\triangle ABC$ . Prove that  $\angle EGF \cong \angle EDF$ .

**Challenge 1** Investigate the case when  $\triangle ABC$  is equilateral.

**Challenge 2** Investigate the case when  $AC = CB$ .



**1-12** In right  $\triangle ABC$ , with right angle at  $C$ ,  $BD = BC$ ,  $AE = AC$ ,  $\overline{EF} \perp \overline{BC}$ , and  $\overline{DG} \perp \overline{AC}$  (Fig. 1-12). Prove that  $DE = EF + DG$ .

**1-13** Prove that the sum of the measures of the perpendiculars from any point on a side of a rectangle to the diagonals is constant.

**Challenge** If the point were on the extension of a side of the rectangle, would the result still hold?

**1-14** The trisectors of the angles of a rectangle are drawn. For each pair of adjacent angles, the trisectors that are closest to the enclosed side are extended until a point of intersection is established. The line segments connecting those points of intersection form a quadrilateral. Prove that the quadrilateral is a rhombus.

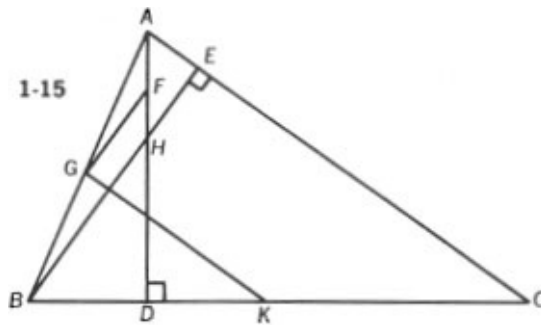
**Challenge 1** What type of quadrilateral would be formed if the original rectangle were replaced by a square?

**Challenge 2** What type of figure is obtained when the original figure is any parallelogram?

**Challenge 3** What type of figure is obtained when the original figure is a rhombus?

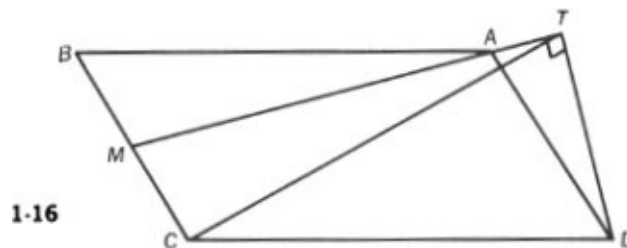
**1-15** In Fig. 1-15,  $\overline{BE}$  and  $\overline{AD}$  are altitudes of  $\triangle ABC$ .  $F$ ,  $G$ , and  $K$  are midpoints of  $\overline{AH}$ ,  $\overline{AB}$ , and  $\overline{BC}$ .

respectively. Prove that  $\angle FGK$  is a right angle.



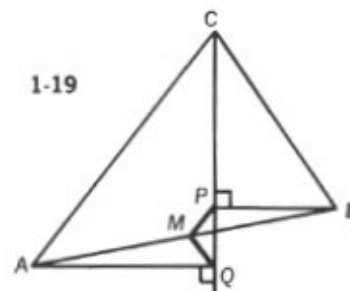
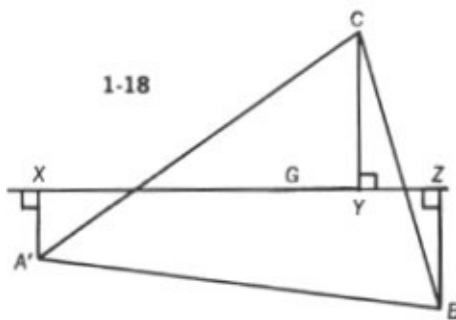
**1-16** In parallelogram  $ABCD$ ,  $M$  is the midpoint of  $\overline{BC}$ .  $\overline{DT}$  is drawn from  $D$  perpendicular to  $\overline{MA}$  in Fig. 1-16. Prove that  $CT = CD$ .

**Challenge** Make the necessary changes in the construction lines, and then prove the theorem for rectangle.



**1-17** Prove that the line segment joining the midpoints of two opposite sides of any quadrilateral bisects the line segment joining the midpoints of the diagonals.

**1-18** In any  $\triangle ABC$ ,  $\overline{XYZ}$  is any line through the centroid  $G$  (Fig. 1-18). Perpendiculars are drawn from each vertex of  $\triangle ABC$  to this line. Prove  $CY = AX + BZ$ .

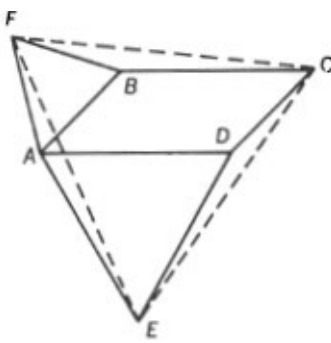


**1-19** In any  $\triangle ABC$ ,  $\overline{CPQ}$  is any line through  $C$ , interior to  $\triangle ABC$  (Fig. 1-19).  $\overline{BP}$  is perpendicular to line  $\overline{CPQ}$ ,  $\overline{AQ}$  is perpendicular to line  $\overline{CPQ}$ , and  $M$  is the midpoint of  $\overline{AB}$ . Prove that  $MP = MQ$ .

**Challenge** Show that the same result holds if the line through  $C$  is exterior to  $\triangle ABC$ .

**1-20** In Fig. 1-20,  $ABCD$  is a parallelogram with equilateral triangles  $ABF$  and  $ADE$  drawn on sides  $\overline{AB}$  and  $\overline{AD}$ , respectively. Prove that  $\triangle FCE$  is equilateral.

1-20



- 1-21** If a square is drawn externally on each side of a parallelogram, prove that
- (a) the quadrilateral determined by the centers of these squares is itself a square
  - (b) the diagonals of the newly formed square are concurrent with the diagonals of the original parallelogram.

**Challenge** Consider other regular polygons drawn externally on the sides of a parallelogram. Study each of these situations!

## 2. Triangles in Proportion

As the title suggests, these problems deal primarily with similarity of triangles. Some interesting geometric proportions are investigated, and there is a geometric illustration of a harmonic mean.

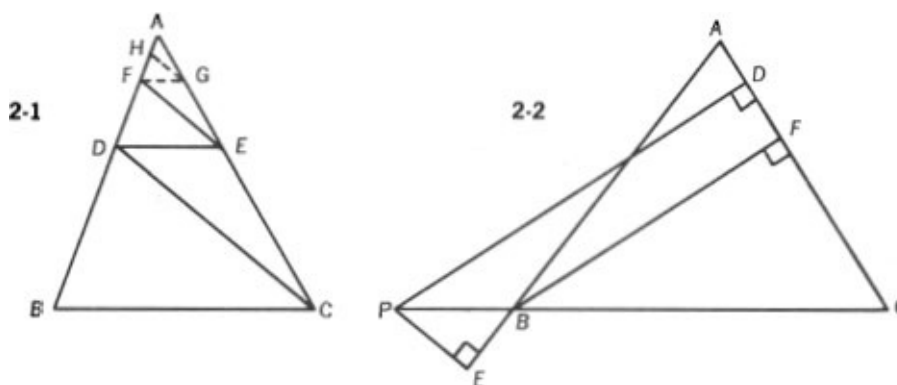
Do you remember manipulations with proportions such as: if  $\frac{a}{b} = \frac{c}{d}$  then  $\frac{a-b}{b} = \frac{c-d}{d}$ ? They are essential to solutions of many problems.

**2-1** In  $\triangle ABC$ ,  $\overline{DE} \parallel \overline{BC}$ ,  $\overline{FE} \parallel \overline{DC}$ ,  $AF = 4$ , and  $FD = 6$  (Fig. 2-1). Find  $DB$ .

**Challenge 1** Find  $DB$  if  $AF = m_1$  and  $FD = m_2$ .

**Challenge 2**  $\overline{FG} \parallel \overline{DE}$ , and  $\overline{HG} \parallel \overline{FE}$ . Find  $DB$  if  $AH = 2$  and  $HF = 4$ .

**Challenge 3** Find  $DB$  if  $AH = m_1$  and  $HF = m_2$ .



**2-2** In isosceles  $\triangle ABC$  ( $AB=AC$ ),  $\overline{CB}$  is extended through  $B$  to  $P$  (Fig. 2-2). A line from  $P$ , parallel to altitude  $\overline{BF}$ , meets  $\overline{AC}$  at  $D$  (where  $D$  is between  $A$  and  $F$ ). From  $P$ , a perpendicular is drawn to meet the extension of  $\overline{AB}$  at  $E$  so that  $B$  is between  $E$  and  $A$ . Express  $BF$  in terms of  $PD$  and  $PE$ . Try solving this problem in two different ways.

**Challenge** Prove that  $BF = PD + PE$  when  $AB = AC$ ,  $P$  is between  $B$  and  $C$ ,  $D$  is between  $C$  and  $A$ , and a perpendicular from  $P$  meets  $\overline{AB}$  at  $E$ .

**2-3** The measure of the longer base of a trapezoid is 97. The measure of the line segment joining the midpoints of the diagonals is 3. Find the measure of the shorter base.

**Challenge** Find a general solution applicable to any trapezoid.

**2-4** In  $\triangle ABC$ ,  $D$  is a point on side  $\overline{BA}$  such that  $BD:DA = 1:2$ .  $E$  is a point on side  $\overline{CB}$  so that  $CE:EB = 1:4$ . Segments  $\overline{DC}$  and  $\overline{AE}$  intersect at  $F$ . Express  $CF:FD$  in terms of two positive relatively prime integers.

**Challenge** Show that if  $BD:DA = m:n$  and  $CE:EB = r:s$ , then

$$\frac{CF}{FD} = \left(\frac{r}{s}\right)\left(\frac{m+n}{n}\right).$$

**2-5** In  $\triangle ABC$ ,  $\overline{BE}$  is a median and  $O$  is the midpoint of  $\overline{BE}$ . Draw  $\overline{AO}$  and extend it to meet  $\overline{BC}$  at  $D$ . Draw  $\overline{CO}$  and extend it to meet  $\overline{BA}$  at  $F$ . If  $CO = 5$ ,  $OF = 5$ , and  $AO = 12$ , find the measure of  $\overline{OD}$ .

**Challenge** Can you establish a relationship between  $OD$  and  $AO$ ?

**2-6** In parallelogram  $ABCD$ , points  $E$  and  $F$  are chosen on diagonal  $\overline{AC}$  so that  $AE = FC$ . If  $\overline{BE}$  extended to meet  $\overline{AD}$  at  $H$ , and  $\overline{BF}$  is extended to meet  $\overline{DC}$  at  $G$ , prove that  $\overline{HG}$  is parallel to  $\overline{AC}$ .

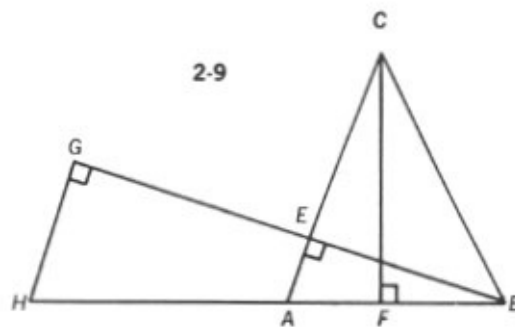
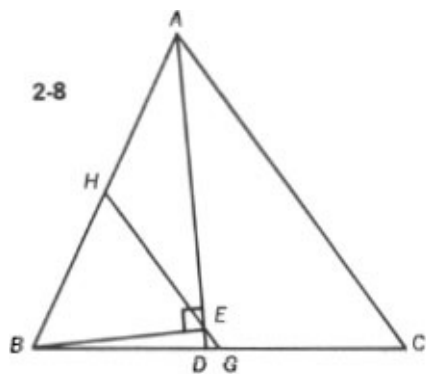
**Challenge** Prove the theorem if  $E$  and  $F$  are on  $\overline{AC}$ , exterior to the parallelogram.

**2-7**  $\overline{AM}$  is the median to side  $\overline{BC}$  of  $\triangle ABC$ , and  $P$  is any point on  $\overline{AM}$ .  $\overline{BP}$  extended meets  $\overline{AC}$  at  $D$  and  $\overline{CP}$  extended meets  $\overline{AB}$  at  $E$ . Prove that  $\overline{DE}$  is parallel to  $\overline{BC}$ .

**Challenge** Show that the result holds if  $P$  is on  $\overline{AM}$ , exterior to  $\triangle ABC$ .

**2-8** In  $\triangle ABC$ , the bisector of  $\angle A$  intersects  $\overline{BC}$  at  $D$  (Fig. 2-8). A perpendicular to  $\overline{AD}$  from  $B$  intersects  $\overline{AD}$  at  $E$ . A line segment through  $E$  and parallel to  $\overline{AC}$  intersects  $\overline{BC}$  at  $G$ , and  $\overline{AB}$  at  $H$ . If  $AB = 26$ ,  $BC = 28$ ,  $AC = 30$ , find the measure of  $\overline{DG}$ .

**Challenge** Prove the result for  $\overline{CF} \perp \overline{AD}$  where  $F$  is on  $\overline{AD}$  exterior to  $\triangle ABC$ .



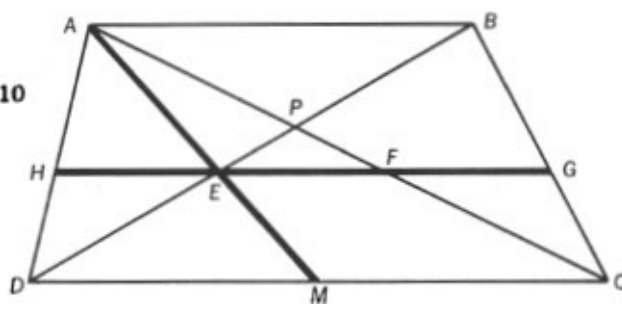
**2-9** In  $\triangle ABC$ , altitude  $\overline{BE}$  is extended to  $G$  so that  $EG =$  the measure of altitude  $\overline{CF}$ . A line through  $G$  and parallel to  $\overline{AC}$  meets  $\overline{BA}$  at  $H$ , as in Fig. 2-9. Prove that  $AH = AC$ .

**Challenge 1** Show that the result holds when  $\angle A$  is a right angle.

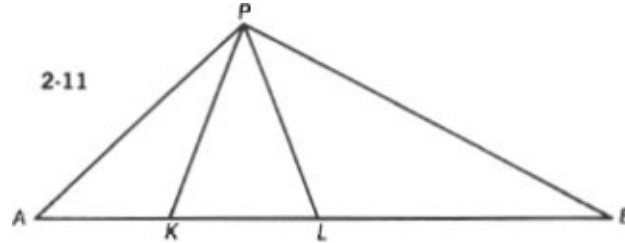
**Challenge 2** Prove the theorem for the case where the measure of altitude  $\overline{BE}$  is greater than the measure of altitude  $\overline{CF}$ , and  $G$  is on  $\overline{BE}$  (between  $B$  and  $E$ ) so that  $EG = CF$ .

**2-10** In trapezoid  $ABCD$  ( $\overline{AB} \parallel \overline{DC}$ ), with diagonals  $\overline{AC}$  and  $\overline{DB}$  intersecting at  $P$ ,  $\overline{AM}$ , a median of  $\triangle ADC$ , intersects  $\overline{BD}$  at  $E$  (Fig. 2-10). Through  $E$ , a line is drawn parallel to  $\overline{DC}$  cutting  $\overline{AC}$ , and  $\overline{BC}$  at points  $H$ ,  $F$ , and  $G$ , respectively. Prove that  $HE = EF = FG$ .

2-10



- 2-11 A line segment  $\overline{AB}$  is divided by points  $K$  and  $L$  in such a way that  $(AL)^2 = (AK)(AB)$  (Fig. 11). A line segment  $\overline{AP}$  is drawn congruent to  $\overline{AL}$ . Prove that  $\overline{PL}$  bisects  $\angle KPB$ .

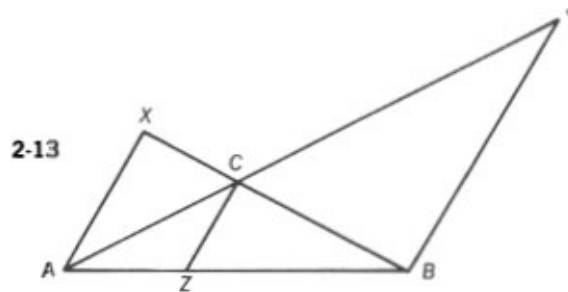


2-11

**Challenge** Investigate the situation when  $\angle APB$  is a right angle.

- 2-12  $P$  is any point on altitude  $\overline{CD}$  of  $\triangle ABC$ .  $\overline{AP}$  and  $\overline{BP}$  meet sides  $\overline{CB}$  and  $\overline{CA}$  at points  $Q$  and  $R$  respectively. Prove that  $\angle QDC \cong \angle RDC$ .

- 2-13 In  $\triangle ABC$ ,  $Z$  is any point on base  $\overline{AB}$  (Fig. 2-13).  $\overline{CZ}$  is drawn. A line is drawn through  $C$  parallel to  $\overline{CZ}$  meeting  $\overline{BC}$  at  $X$ . A line is drawn through  $B$  parallel to  $\overline{CZ}$  meeting  $\overline{AC}$  at  $Y$ . Prove that  $\frac{1}{AX} + \frac{1}{BY} = \frac{1}{CZ}$ .



2-13

**Challenge** Two telephone cable poles, 40 feet and 60 feet high, respectively, are placed near each other. As partial support, a line runs from the top of each pole to the bottom of the other. How high above the ground is the point of intersection of the two support lines?

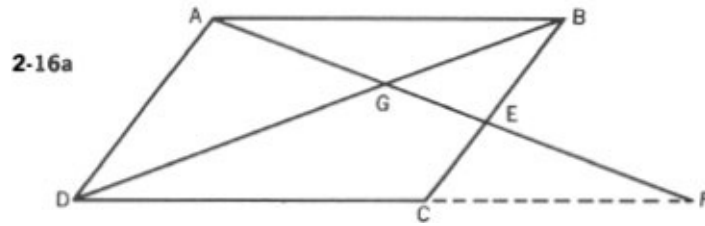
- 2-14 In  $\triangle ABC$ ,  $m\angle A = 120$ . Express the measure of the internal bisector of  $\angle A$  in terms of the two adjacent sides.

**Challenge** Prove the converse of the theorem established above.

- 2-15 Prove that the measure of the segment passing through the point of intersection of the diagonals of a trapezoid and parallel to the bases with its endpoints on the legs, is the harmonic mean between the measures of the parallel sides. The harmonic mean of two numbers is defined as the reciprocal of the average of the reciprocals of two numbers. The harmonic mean between  $a$  and  $b$  is equal to

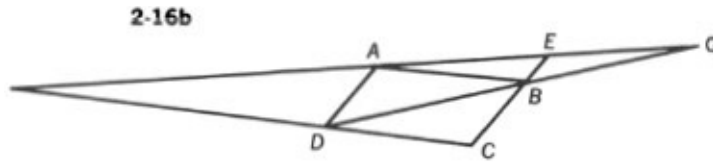
$$\left(\frac{a^{-1} + b^{-1}}{2}\right)^{-1} = \frac{2ab}{a + b}.$$

**2-16** In  $\square ABCD$ ,  $E$  is on  $\overline{BC}$  (Fig. 2-16a).  $\overline{AE}$  cuts diagonal  $\overline{BD}$  at  $G$  and  $\overline{DC}$  at  $F$ . If  $AG = 6$  and  $GE = 4$ , find  $EF$ .



**Challenge 1** Show that  $AG$  is one-half the harmonic mean between  $AF$  and  $AE$ .

**Challenge 2** Prove the theorem when  $E$  is on the extension of  $\overline{CB}$  through  $B$  (Fig. 2-16b).

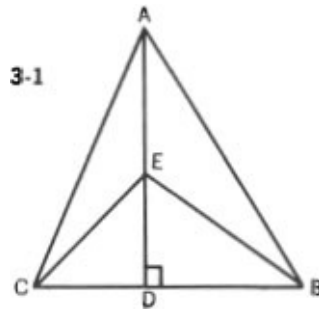




### 3. The Pythagorean Theorem

You will find two kinds of problems in this section concerning the key result of Euclidean geometry, the theorem of Pythagoras. Some problems involve direct applications of the theorem. Others make use of results that depend on the theorem, such as the relationship between the sides of an isosceles right triangle or a 30–60–90 triangle.

**3-1** In any  $\triangle ABC$ ,  $E$  is any point on altitude  $\overline{AD}$  (Fig. 3-1). Prove that  $(AC)^2 - (CE)^2 = (AB)^2 - (EB)^2$ .



**Challenge 1** Show that the result holds if  $E$  is on the extension of  $\overline{AD}$  through  $D$ .

**Challenge 2** What change in the theorem results if  $E$  is on the extension of  $\overline{AD}$  through  $A$ ?

**3-2** In  $\triangle ABC$ , median  $\overline{AD}$  is perpendicular to median  $\overline{BE}$ . Find  $AB$  if  $BC = 6$  and  $AC = 8$ .

**Challenge 1** Express  $AB$  in general terms for  $BC = a$ , and  $AC = b$ .

**Challenge 2** Find the ratio of  $AB$  to the measure of its median.

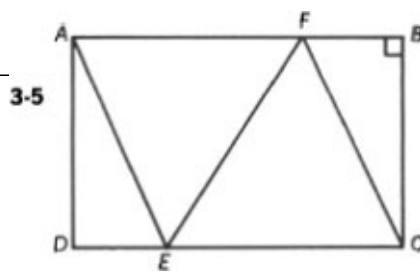
**3-3** On hypotenuse  $\overline{AB}$  of right  $\triangle ABC$ , draw square  $ABLH$  externally. If  $AC = 6$  and  $BC = 8$ , find  $CH$ .

**Challenge 1** Find the area of quadrilateral  $HLBC$ .

**Challenge 2** Solve the problem if square  $ABLH$  overlaps  $\triangle ABC$ .

**3-4** The measures of the sides of a right triangle are 60, 80, and 100. Find the measure of a line segment, drawn from the vertex of the right angle to the hypotenuse, that divides the triangle into two triangles of equal perimeters.

**3-5** On sides  $\overline{AB}$  and  $\overline{DC}$  of rectangle  $ABCD$ , points  $F$  and  $E$  are chosen so that  $AFCE$  is a rhombus (Fig. 3-5). If  $AB = 16$  and  $BC = 12$ , find  $EF$ .



**Challenge** If  $AB = a$  and  $BC = b$ , what general expression will give the measure of  $\overline{EF}$ ?

**3-6** A man walks one mile east, then one mile northeast, then another mile east. Find the distance in miles, between the man's initial and final positions.

**Challenge** How much shorter (or longer is the distance if the course is one mile east, one mile north, then one mile east?

**3-7** If the measures of two sides and the included angle of a triangle are  $7$ ,  $\sqrt{50}$ , and  $135^\circ$  respectively, find the measure of the segment joining the midpoints of the two given sides.

**Challenge 1** Show that when  $m\angle A = 135^\circ$ ,

$$EF = \frac{1}{2} \sqrt{b^2 + c^2 + bc\sqrt{2}},$$

where  $E$  and  $F$  are midpoints of sides  $\overline{AC}$  and  $\overline{AB}$ , respectively, of  $\triangle ABC$ .

NOTE:  $a$ ,  $b$ , and  $c$  are the lengths of the sides opposite  $\angle A$ ,  $\angle B$ , and  $\angle C$  of  $\triangle ABC$ .

**Challenge 2** Show that when  $m\angle A = 120^\circ$ ,

$$EF = \frac{1}{2} \sqrt{b^2 + c^2 + bc\sqrt{1}}.$$

**Challenge 3** Show that when  $m\angle A = 150^\circ$ ,

$$EF = \frac{1}{2} \sqrt{b^2 + c^2 + bc\sqrt{3}}.$$

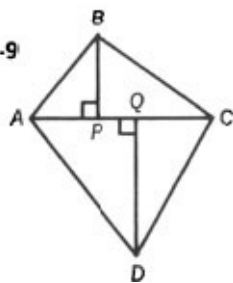
**Challenge 4** On the basis of these results, predict the values of  $EF$  for  $m\angle A = 30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$ .

**3-8** Hypotenuse  $\overline{AB}$  of right  $\triangle ABC$  is divided into four congruent segments by points  $G$ ,  $E$ , and  $H$  in the order  $A, G, E, H, B$ . If  $AB = 20$ , find the sum of the squares of the measures of the line segments from  $C$  to  $G$ ,  $E$ , and  $H$ .

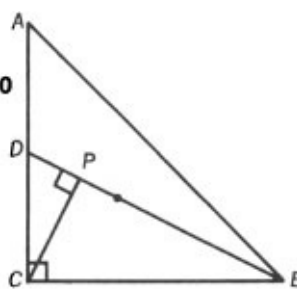
**Challenge** Express the result in general terms when  $AB = c$ .

**3-9** In quadrilateral  $ABCD$ ,  $AB = 9$ ,  $BC = 12$ ,  $CD = 13$ ,  $DA = 14$ , and diagonal  $AC = 15$  (Fig. 3-9). Perpendiculars are drawn from  $B$  and  $D$  to  $AC$ , meeting  $AC$  at points  $P$  and  $Q$ , respectively. Find  $PQ$ .

3-9



3-10



**3-10** In  $\triangle ABC$ , angle  $C$  is a right angle (Fig. 3-10).  $AC = BC = 1$ , and  $D$  is the midpoint of  $\overline{AB}$ .  $\overline{CD}$  is drawn, and a line perpendicular to  $\overline{BD}$  at  $P$  is drawn from  $C$ . Find the distance from  $P$  to the intersection of the medians of  $\triangle ABC$ .

**Challenge** Show that  $PG = \frac{c\sqrt{10}}{30}$ , when  $G$  is the centroid, and  $c$  is the length of the hypotenuse.

**3-11** A right triangle contains a  $60^\circ$  angle. If the measure of the hypotenuse is 4, find the distance from the point of intersection of the 2 legs of the triangle to the point of intersection of the angle bisectors.

**3-12** From point  $P$  inside  $\triangle ABC$ , perpendiculars are drawn to the sides meeting  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$ , points  $D$ ,  $E$ , and  $F$ , respectively. If  $BD = 8$ ,  $DC = 14$ ,  $CE = 13$ ,  $AF = 12$ , and  $FB = 6$ , find  $AP$ . Derive a general theorem, and then make use of it to solve this problem.

**3-13** For  $\triangle ABC$  with medians  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$ , let  $m = AD + BE + CF$ , and let  $s = AB + BC + CA$ . Prove that  $\frac{3}{2}s > m > \frac{3}{4}s$ .

**3-14** Prove that  $\frac{3}{4}(a^2 + b^2 + c^2) = m_a^2 + m_b^2 + m_c^2$ . ( $m_c$  means the measure of the median drawn to side  $c$ .)

**Challenge 1** Verify this relation for an equilateral triangle.

**Challenge 2** The sum of the squares of the measures of the sides of a triangle is 120. If two of the medians measure 4 and 5, respectively, how long is the third median?

**Challenge 3** If  $\overline{AE}$  and  $\overline{BF}$  are medians drawn to the legs of right  $\triangle ABC$ , find the numeral value of  $\frac{(AE)^2 + (BF)^2}{(AB)^2}$ .

## 4. Circles Revisited

Circles are the order of the day in this section. There are problems dealing with arc and angle measurement; others deal with lengths of chords, secants, tangents, and radii; and some problems involve both.

Particular attention should be given to [Problems 4-33](#) thru [4-40](#), which concern cyclic quadrilaterals (quadrilaterals that may be inscribed in a circle). This often neglected subject has interesting applications. If you are not familiar with it, you might look at the theorems that are listed in [Appendix I](#).

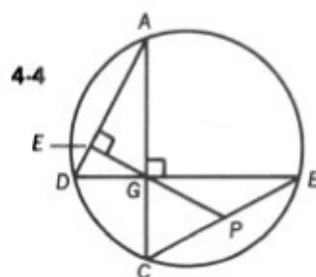
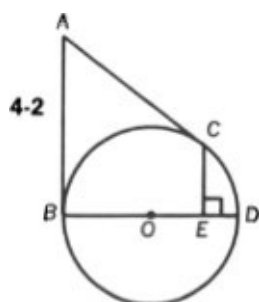
**4-1** Two tangents from an external point  $P$  are drawn to a circle, meeting the circle at points  $A$  and  $B$ . A third tangent meets the circle at  $T$ , and tangents  $\overline{PA}$  and  $\overline{PB}$  at points  $Q$  and  $R$ , respectively. Find the perimeter  $p$  of  $\triangle PQR$ .

**4-2**  $\overline{AB}$  and  $\overline{AC}$  are tangent to circle  $O$  at  $B$  and  $C$ , respectively, and  $\overline{CE}$  is perpendicular to diameter  $\overline{BD}$  (Fig. 4-2). Prove  $(BE)(BO) = (AB)(CE)$ .

**Challenge 1** Find the value of  $AB$  when  $E$  coincides with  $O$ .

**Challenge 2** Show that the theorem is true when  $E$  is between  $B$  and  $O$ .

**Challenge 3** Show that  $\frac{AB}{\sqrt{BE}} = \frac{BO}{\sqrt{ED}}$ .



**4-3** From an external point  $P$ , tangents  $\overline{PA}$  and  $\overline{PB}$  are drawn to a circle. From a point  $Q$  on the major (or minor) arc  $\widehat{AB}$ , perpendiculars are drawn to  $\overline{AB}$ ,  $\overline{PA}$ , and  $\overline{PB}$ . Prove that the perpendicular to  $\overline{AB}$  is the mean proportional between the other two perpendiculars.

**Challenge** Show that the theorem is true when the tangents are parallel.

**4-4** Chords  $\overline{AC}$  and  $\overline{DB}$  are perpendicular to each other and intersect at point  $G$  (Fig. 4-4). In  $\triangle AGD$  the altitude from  $G$  meets  $\overline{AD}$  at  $E$ , and when extended meets  $\overline{BC}$  at  $P$ . Prove that  $BP = PC$ .

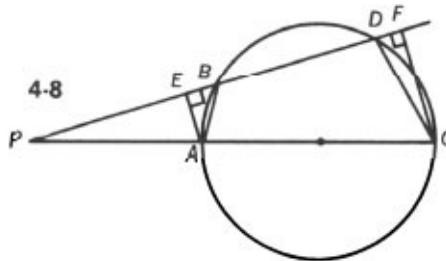
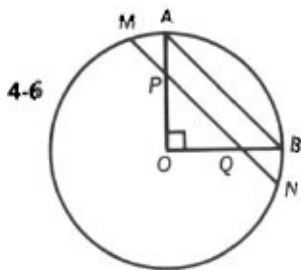
**Challenge** One converse of this theorem is as follows. Chords  $\overline{AC}$  and  $\overline{DB}$  intersect at  $G$ . In  $\triangle AGD$  the altitude from  $G$  meets  $\overline{AD}$  at  $E$ , and when extended meets  $\overline{BC}$  at  $P$  so that  $BP = PC$ . Prove that  $\overline{AC} \perp \overline{BD}$ .

**4-5** Square  $ABCD$  is inscribed in a circle. Point  $E$  is on the circle. If  $AB = 8$ , find the value of

$$(AE)^2 + (BE)^2 + (CE)^2 + (DE)^2.$$

**Challenge** Prove that for  $ABCD$ , a non-square rectangle,  $(AE)^2 + (BE)^2 + (CE)^2 + (DE)^2 = 2d^2$  where  $d$  is the measure of the length of a diagonal of the rectangle.

**4-6** Radius  $\overline{AO}$  is perpendicular to radius  $\overline{OB}$ ,  $\overline{MN}$  is parallel to  $\overline{AB}$  meeting  $\overline{AO}$  at  $P$  and  $\overline{OB}$  at  $Q$  and the circle at  $M$  and  $N$  (Fig. 4-6). If  $MP = \sqrt{56}$ , and  $PN = 12$ , find the measure of the radius of the circle.



**4-7** Chord  $\overline{CD}$  is drawn so that its midpoint is 3 inches from the center of a circle with a radius 6 inches. From  $A$ , the midpoint of minor arc  $\overline{CD}$ , any chord  $\overline{AB}$  is drawn intersecting  $\overline{CD}$  in  $M$ . Let  $v$  be the range of values of  $(AB)(AM)$ , as chord  $\overline{AB}$  is made to rotate in the circle about the fixed point  $A$ . Find  $v$ .

**4-8** A circle with diameter  $\overline{AC}$  is intersected by a secant at points  $B$  and  $D$ . The secant and the diameter intersect at point  $P$  outside the circle, as shown in Fig. 4-8. Perpendiculars  $\overline{AE}$  and  $\overline{CF}$  are drawn from the extremities of the diameter to the secant. If  $EB = 2$ , and  $BD = 6$ , find  $DF$ .

**Challenge** Does  $DF = EB$ ? Prove it!

**4-9** A diameter  $\overline{CD}$  of a circle is extended through  $D$  to external point  $P$ . The measure of secant  $\overline{CP}$  is 77. From  $P$ , another secant is drawn intersecting the circle first at  $A$ , then at  $B$ . The measure of secant  $\overline{PB}$  is 33. The diameter of the circle measures 74. Find the measure of the angle formed by the secants.

**Challenge** Find the measure of the shorter secant if the measure of the angle between the secants is 45.

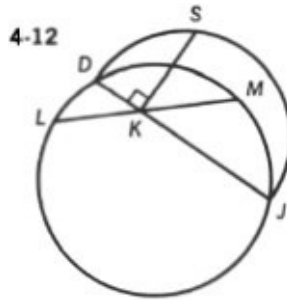
**4-10** In  $\triangle ABC$ , in which  $AB = 12$ ,  $BC = 18$ , and  $AC = 25$ , a semicircle is drawn so that its diameter lies on  $\overline{AC}$ , and so that it is tangent to  $\overline{AB}$  and  $\overline{BC}$ . If  $O$  is the center of the circle, find the measure of  $\overline{AO}$ .

**Challenge** Find the diameter of the semicircle.

**4-11** Two parallel tangents to circle  $O$  meet the circle at points  $M$  and  $N$ . A third tangent to circle  $O$  at point  $P$ , meets the other two tangents at points  $K$  and  $L$ . Prove that a circle, whose diameter is  $\overline{KL}$ , passes through  $O$ , the center of the original circle.

**Challenge** Prove that for different positions of point  $P$ , on  $\overline{MN}$ , a family of circles is obtained, all of which are tangent to each other at  $O$ .

**4-12**  $\overline{LM}$  is a chord of a circle, and is bisected at  $K$  (Fig. 4-12).  $\overline{DKJ}$  is another chord. A semicircle is drawn with diameter  $\overline{DJ}$ .  $\overline{KS}$ , perpendicular to  $\overline{DJ}$ , meets this semicircle at  $S$ . Prove  $KS = KL$ .



**4-13**  $\triangle ABC$  is inscribed in a circle with diameter  $\overline{AD}$ . A tangent to the circle at  $D$  cuts  $\overline{AB}$  extended at  $E$  and  $\overline{AC}$  extended at  $F$ . If  $AB = 4$ ,  $AC = 6$ , and  $BE = 8$ , find  $CF$ .

**Challenge 1** Find  $m\angle DAF$ .

**Challenge 2** Find  $BC$ .

**4-14** Altitude  $\overline{AD}$  of equilateral  $\triangle ABC$  is a diameter of circle  $O$ . If the circle intersects  $\overline{AB}$  and  $\overline{AC}$  at  $E$  and  $F$ , respectively, find the ratio of  $EF : BC$ .

**Challenge** Find the ratio of  $EB : BD$ .

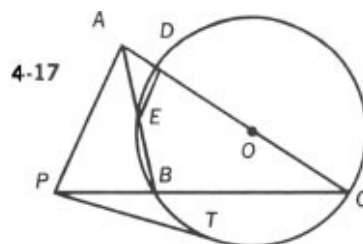
**4-15** Two circles intersect in  $A$  and  $B$ , and the measure of the common chord  $\overline{AB}$  is 10. The line joining the centers cuts the circles in  $P$  and  $Q$ . If  $PQ = 3$  and the measure of the radius of one circle is 13, find the radius of the other circle.

**Challenge** Find the second radius if  $PQ = 2$ .

**4-16**  $ABCD$  is a quadrilateral inscribed in a circle. Diagonal  $\overline{BD}$  bisects  $\overline{AC}$ . If  $AB = 10$ ,  $AD = 12$ , and  $DC = 11$ , find  $BC$ .

**Challenge** Solve the problem when diagonal  $\overline{BD}$  divides  $\overline{AC}$  into two segments, one of which is twice as long as the other.

**4-17**  $A$  is a point exterior to circle  $O$ .  $\overline{PT}$  is drawn tangent to the circle so that  $PT = PA$ . As shown in Fig. 4-17,  $C$  is any point on circle  $O$ , and  $\overline{AC}$  and  $\overline{PC}$  intersect the circle at points  $D$  and  $E$ , respectively.  $\overline{AB}$  intersects the circle at  $E$ . Prove that  $\overline{DE}$  is parallel to  $\overline{AP}$ .



**Challenge 1** Prove the theorem for  $A$  interior to circle  $O$ .

**Challenge 2** Explain the situation when  $A$  is on circle  $O$ .

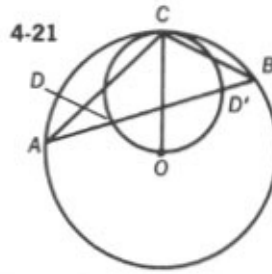
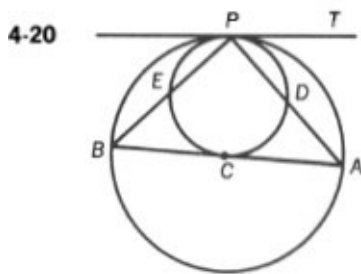
**4-18**  $\overline{PA}$  and  $\overline{PB}$  are tangents to a circle, and  $\overline{PCD}$  is a secant. Chords  $\overline{AC}$ ,  $\overline{BC}$ ,  $\overline{BD}$ , and  $\overline{DA}$  are drawn. If  $AC = 9$ ,  $AD = 12$ , and  $BD = 10$ , find  $BC$ .

**Challenge** If in addition to the information given above,  $PA = 15$  and  $PC = 9$ , find  $AB$ .

**4-19** The altitudes of  $\triangle ABC$  meet at  $O$ .  $\overline{BC}$ , the base of the triangle, has a measure of 16. The circumcircle of  $\triangle ABC$  has a diameter with a measure of 20. Find  $AO$ .

**4-20** Two circles are tangent internally at  $P$ , and a chord  $\overline{AB}$  of the larger circle is tangent to the smaller circle at  $C$  (Fig. 4-20).  $\overline{PB}$  and  $\overline{PA}$  cut the smaller circle at  $E$  and  $D$ , respectively. If  $AB = 15$ , while  $PE = 2$  and  $PD = 3$ , find  $AC$ .

**Challenge** Express  $AC$  in terms of  $AB$ ,  $PE$ , and  $PD$ .



**4-21** A circle, center  $O$ , is circumscribed about  $\triangle ABC$ , a triangle in which  $\angle C$  is obtuse (Fig. 4-21). With  $\overline{OC}$  as diameter, a circle is drawn intersecting  $\overline{AB}$  in  $D$  and  $D'$ . If  $AD = 3$ , and  $DB = 4$ , find  $CD$ .

**Challenge 1** Show that the theorem is or is not true if  $m\angle C = 90$ .

**Challenge 2** Investigate the case for  $m\angle C < 90$ .

**4-22** In circle  $O$ , perpendicular chords  $\overline{AB}$  and  $\overline{CD}$  intersect at  $E$  so that  $AE = 2$ ,  $EB = 12$ , and  $CE = 4$ . Find the measure of the radius of circle  $O$ .

**Challenge** Find the shortest distance from  $E$  to the circle.

**4-23** Prove that the sum of the measure of the squares of the segments made by two perpendicular chords is equal to the square of the measure of the diameter of the given circle.

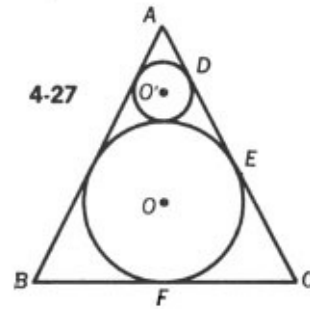
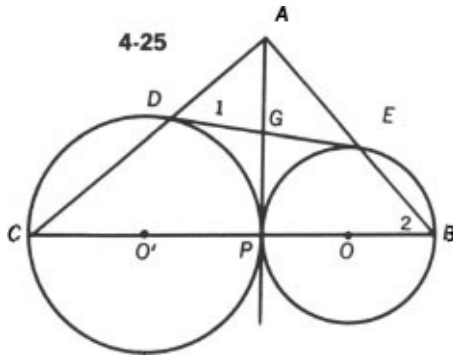
**Challenge** Prove the theorem for two perpendicular chords meeting outside the circle.

**4-24** Two equal circles are tangent externally at  $T$ . Chord  $\overline{TM}$  in circle  $O$  is perpendicular to chord  $\overline{TN}$  in circle  $Q$ . Prove that  $\overline{MN} \parallel \overline{OQ}$  and  $MN = OQ$ .

**Challenge** Show that  $MN = \sqrt{2(R^2 + r^2)}$  if the circles are unequal, where  $R$  and  $r$  are the radii of the two circles.

**4-25** From point  $A$  on the common internal tangent of tangent circles  $O$  and  $O'$ , secants  $\overline{AEB}$  and  $\overline{ADC}$  are drawn, respectively (Fig. 4-25). If  $\overline{DC}$  is the common external tangent, and points  $C$  and  $B$  are collinear with the centers of the circles, prove

- (a)  $m\angle 1 = m\angle 2$ , and
- (b)  $\angle A$  is a right angle.



**Challenge 1** Prove or disprove that if  $\overline{BC}$  does not pass through the centers of the circles, the designated pairs of angles are not equal and  $\angle A$  is not a right angle.

**Challenge 2** Prove that  $DE$  is the mean proportional between the diameters of circles  $O$  and  $O'$ .

**4-26** Two equal intersecting circles  $O$  and  $O'$  have a common chord  $\overline{RS}$ . From any point  $P$  on  $\overline{RS}$ , a ray is drawn perpendicular to  $\overline{RS}$  cutting circles  $O$  and  $O'$  at  $A$  and  $B$ , respectively. Prove that  $\overline{AB}$  is parallel to the line of centers,  $\overline{OO'}$ , and that  $AB = OO'$ .

**4-27** A circle is inscribed in a triangle whose sides are 10, 10, and 12 units in measure (Fig. 4-27). A second, smaller circle is inscribed tangent to the first circle and to the equal sides of the triangle. Find the measure of the radius of the second circle.

**Challenge 1** Solve the problem in general terms if  $AC = a$ ,  $BC = 2b$ .

**Challenge 2** Inscribe a third, smaller circle tangent to the second circle and to the equal sides, and find its radius by inspection.

**Challenge 3** Extend the legs of the triangle through  $B$  and  $C$ , and draw a circle tangent to the original circle and to the extensions of the legs. What is its radius?

**4-28** A circle with radius 3 is inscribed in a square. Find the radius of the circle that is inscribed between two sides of the square and the original circle.

**Challenge** Show that the area of the small circle is approximately 3% of the area of the large circle.

**4-29**  $\overline{AB}$  is a diameter of circle  $O$ , as shown in Fig. 4-29. Two circles are drawn with  $\overline{AO}$  and  $\overline{OB}$  as diameters. In the region between the circumferences, a circle  $D$  is inscribed, tangent to the three previous circles. If the measure of the radius of circle  $D$  is 8, find  $AB$ .

**Challenge** Prove that the area of the shaded region equals the area of circle  $E$ .



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