Challenging Problems in Geometry

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Bibliographical Note

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INTRODUCTION

The challenge of well-posed problems transcends national boundaries, ethnic origins, political systems, economic doctrines, and religious beliefs; the appeal is almost universal. Why? You are invited to formulate your own explanation. We simply accept the observation and exploit it here for entertainment and enrichment.

This book is a new, combined edition of two volumes first published in 1970. It contains nearly two hundred problems, many with extensions or variations that we call *challenges*. Supplied with pencil and paper and fortified with a diligent attitude, you can make this material the starting point for exploring unfamiliar or little-known aspects of mathematics. The challenges will spur you on; perhaps you can even supply your own challenges in some cases. A study of these nonroutine problems can provide valuable underpinnings for work in more advanced mathematics.

This book, with slight modifications made, is as appropriate now as it was a quarter century ago when it was first published. The National Council of Teachers of Mathematics (NCTM), in their *Curriculum and Evaluation Standards for High School Mathematics* (1989), lists problem solving as its first standard, stating that "mathematical problem solving in its broadest sense is nearly synonymous with doing mathematics." They go on to say, "[problem solving] is a process by which the fabric of mathematics is identified in later standards as both constructive and reinforced."

This strong emphasis on mathematics is by no means a new agenda item. In 1980, the NCTM published *An Agenda for Action*. There, the NCTM also had problem solving as its first item, stating "educators should give priority to the identification and analysis of specific problem solving strategies …. [and] should develop and disseminate examples of 'good problems' and strategies." It is our intention to provide secondary mathematics educators with materials to help them implement this very important recommendation.

ABOUT THE BOOK

Challenging Problems in Geometry is organized into three main parts: "Problems," "Solutions," and "Hints." Unlike many contemporary problem-solving resources, this book is arranged *not* by problem-solving technique, but by topic. We feel that announcing the technique to be used stifles creativity and destroys a good part of the fun of problem solving.

The problems themselves are grouped into two sections. Section I, "A New Twist on Familiar Topics," covers five topics that roughly parallel the sequence of the high school geometry course. Section II, "Further Investigations," presents topics not generally covered in the high school geometry course, but certainly within the scope of that audience. These topics lead to some very interesting extensions and enable the reader to investigate numerous fascinating geometric relationships.

Within each topic, the problems are arranged in approximate order of difficulty. For some problems, the basic difficulty may lie in making the distinction between relevant and irrelevant data or between known and unknown information. The sure ability to make these distinctions is part of the process of problem solving, and each devotee must develop this power by him- or herself. It will come with sustained effort.

In the "Solutions" part of the book, each problem is restated and then its solution is given. Answers are also provided for many but not all of the challenges. In the solutions (and later in the hints), you will notice citations such as "(#23)" and "(Formula #5b)." These refer to the definitions postulates, and theorems listed in Appendix I, and the formulas given in Appendix II.

~~From time to time we give alternate methods of solution, for there is rarely only one way to~~ solv a problem. The solutions shown are far from exhaustive, and intentionally so, allowing you to try variety of different approaches. Particularly enlightening is the strategy of using multiple method integrating algebra, geometry, and trigonometry. Instances of multiple methods or multip interpretations appear in the solutions. Our continuing challenge to you, the reader, is to find different method of solution for every problem.

The third part of the book, “Hints,” offers suggestions for each problem and for selecte challenges. Without giving away the solution, these hints can help you get back on the track if you ru into difficulty.

USING THE BOOK

This book may be used in a variety of ways. It is a valuable supplement to the basic geomet textbook, both for further explorations on specific topics and for practice in developing problem solving techniques. The book also has a natural place in preparing individuals or student teams f participation in mathematics contests. Mathematics clubs might use this book as a source independent projects or activities. Whatever the use, experience has shown that these problem motivate people of all ages to pursue more vigorously the study of mathematics.

Very near the completion of the first phase of this project, the passing of Professor Charles T Salkind grieved the many who knew and respected him. He dedicated much of his life to the study problem posing and problem solving and to projects aimed at making problem solving meaningfu interesting, and instructive to mathematics students at all levels. His efforts were praised by a Working closely with this truly great man was a fascinating and pleasurable experience.

Alfred S. Posamenti
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PREPARING TO SOLVE A PROBLEM

A strategy for attacking a problem is frequently dictated by the use of analogy. In fact, searching fo an analogue appears to be a psychological necessity. However, some analogues are more apparent tha real, so analogies should be scrutinized with care. Allied to analogy is structural similarity or patter Identifying a pattern in apparently unrelated problems is not a common achievement, but when don successfully it brings immense satisfaction.

Failure to solve a problem is sometimes the result of fixed habits of thought, that is, inflexib approaches. When familiar approaches prove fruitless, be prepared to alter the line of attack. flexible attitude may help you to avoid needless frustration.

Here are three ways to make a problem yield dividends:

(1) The result of formal manipulation, that is, “the answer,” may or may not be meaningful; fin out! Investigate the possibility that the answer is not unique. If more than one answer is obtaine decide on the acceptability of each alternative. Where appropriate, estimate the answer advance of the solution. The habit of estimating in advance should help to prevent crude errors manipulation.

(2) Check possible restrictions on the data and/or the results. Vary the data in significant ways an study the effect of such variations on the original result.

(3) The insight needed to solve a generalized problem is sometimes gained by first specializing Conversely, a specialized problem, difficult when tackled directly, sometimes yields to an eas solution by first generalizing it.

As is often true, there may be more than one way to solve a problem. There is usually what w will refer to as the “peasant’s way” in contrast to the “poet’s way”—the latter being the more elegan method.

To better understand this distinction, let us consider the following problem:

> If the sum of two numbers is 2, and the product of these same two numbers is 3, find the sum of the reciprocals of these two numbers.

Those attempting to solve the following pair of equations simultaneously are embarking on th “peasant’s way” to solve this problem.

$$\begin{aligned} x + y &= 2 \\ xy &= 3 \end{aligned}$$

Substituting for y in the second equation yields the quadratic equation, $x^2 - 2x + 3 = 0$. Using th quadratic formula we can find $x = 1 \pm i\sqrt{2}$. By adding the reciprocals of these two values of x, the answ $\frac{2}{3}$ appears. This is clearly a rather laborious procedure, not particularly elegant.

The “poet’s way” involves working backwards. By considering the desired result

$$\frac{1}{x} + \frac{1}{y}$$

and seeking an expression from which this sum may be derived, one should inspect the algebraic sum

$$\frac{x + y}{xy}$$

The answer to the original problem is now obvious! That is, since $x + y = 2$ and $xy = 3$, $\frac{x + y}{xy} = \frac{2}{3}$. This clearly a more elegant solution than the first one.

The "poet's way" solution to this problem points out a very useful and all too often neglecte method of solution. A reverse strategy is certainly not new. It was considered by Pappus of Alexandri about 320 A.D. In Book VII of Pappus' *Collection* there is a rather complete description of th methods of "analysis" and "synthesis." T. L. Heath, in his book *A Manual of Greek Mathemati* (Oxford University Press, 1931, pp. 452-53), provides a translation of Pappus' definitions of thes terms:

> *Analysis* takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were already done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until, by so retracing our steps, we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backward.
>
> But in *synthesis*, reversing the progress, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of that which was sought: and this we call *synthesis*.

Unfortunately, this method has not received its due emphasis in the mathematics classroom. W hope that the strategy recalled here will serve you well in solving some of the problems presented this book.

Naturally, there are numerous other clever problem-solving strategies to pick from. In rece years a plethora of books describing various problem-solving methods have become available. concise description of these problem-solving strategies can be found in *Teaching Secondary Scho Mathematics: Techniques and Enrichment Units*, by A. S. Posamentier and J. Stepelman, 4th editio (Columbus, Ohio: Prentice Hall/Merrill, 1995).

Our aim in this book is to strengthen the reader's problem-solving skills through nonrouti motivational examples. We therefore allow the reader the fun of finding the best path to a problem solution, an achievement generating the most pleasure in mathematics.

PROBLEMS

SECTION
A New Twist on Familiar Topic

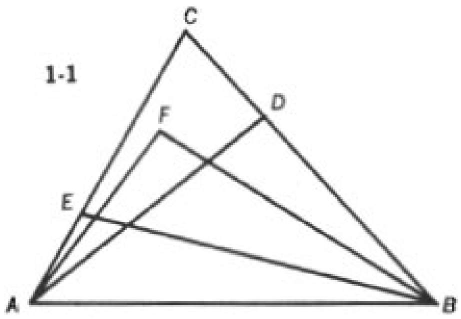
1. Congruence and Parallelism

The problems in this section present applications of several topics that are encountered early in th formal development of plane Euclidean geometry. The major topics are congruence of line segment angles, and triangles and parallelism in triangles and various types of quadrilaterals.

1-1 In any ΔABC, E and D are interior points of $\overline{AC}$ and $\overline{BC}$, respectively (Fig. 1-1). $\overline{AF}$ bisects $\angle CA$ and $\overline{BF}$ bisects $\angle CBE$. Prove $m\angle AEB + m\angle ADB = 2m\angle AFB$.

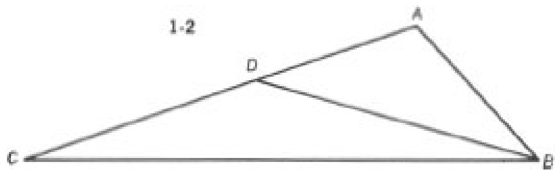
Challenge 1 Prove that this result holds if E coincides with C.

Challenge 2 Prove that the result holds if E and D are exterior points on extensions of $\overline{AC}$ and through C.



1-1

1-2 In ΔABC, a point D is on $\overline{AC}$ so that $AB = AD$ (Fig. 1-2). $m\angle ABC - m\angle ACB = 30$. Find $m\angle CB$

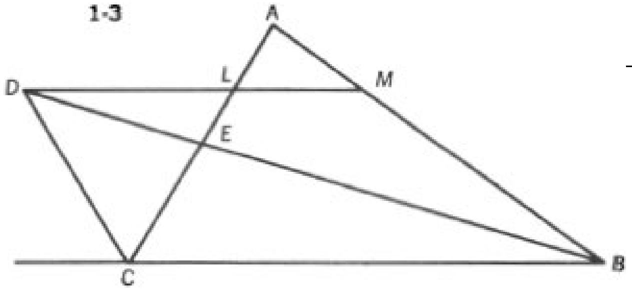


1-2

1-3 The interior bisector of $\angle B$, and the exterior bisector of $\angle C$ of ΔABC meet at D (Fig. 1-3 Through D, a line parallel to $\overline{CB}$ meets $\overline{AC}$ at L and $\overline{AB}$ at M. If the measures of legs $\overline{LC}$ ar $\overline{MB}$ of trapezoid $CLMB$ are 5 and 7, respectively, find the measure of base $\overline{LM}$. Prove yo result.

Challenge Find $\overline{LM}$ if ΔABC is equilateral.

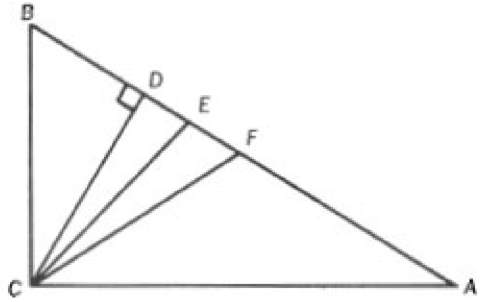
1-3



1-4 In right ΔABC, $\overline{CF}$ is the median to hypotenuse $\overline{AB}$, $\overline{CE}$ is the bisector of $\angle ACB$, and $\overline{CD}$ is th altitude to $\overline{AB}$ (Fig. 1-4). Prove that $\angle DCE \cong \angle ECF$.

Challenge Does this result hold for a non-right triangle?

1-4

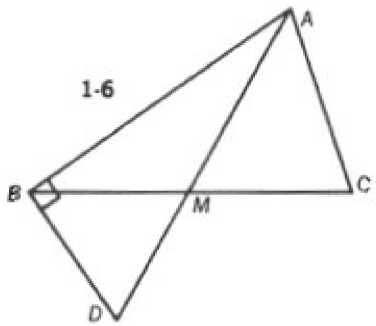


1-5 The measure of a line segment $\overline{PC}$, perpendicular to hypotenuse $\overline{AC}$ of right ΔABC, is equal the measure of leg $\overline{BC}$. Show $\overline{BP}$ may be perpendicular or parallel to the bisector of $\angle A$.

1-6 Prove the following: if, in ΔABC, median $\overline{AM}$ is such that $m\angle BAC$ is divided in the ratio 1: and $\overline{AM}$ is extended through M to D so that $\angle DBA$ is a right angle, then $AC = \frac{1}{2} AD$ (Fig. 1-6).

Challenge Find two ways of proving the theorem when $m\angle A = 90$.

1-6



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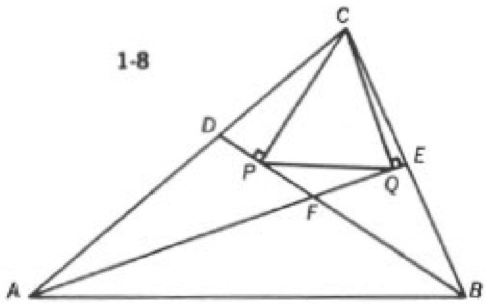


1-7 In square $ABCD$, M is the midpoint of $\overline{AB}$. A line perpendicular to $\overline{MC}$ at M meets $\overline{AD}$ at Prove that $\angle BCM \cong \angle KCM$.

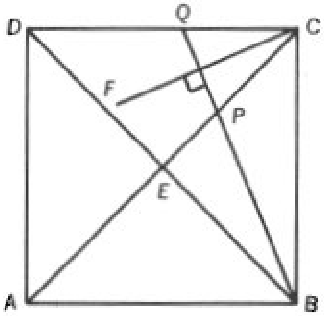
Challenge Prove that ΔKDC is a 3–4–5 right triangle.

1-8 Given any ΔABC, $\overline{AE}$ bisects $\angle BAC$, $\overline{BD}$ bisects $\angle ABC$, $\overline{CP} \perp \overline{BD}$, and $\overline{CQ} \perp \overline{AE}$ (Fig. 1-8), prove th $\overline{PQ}$ is parallel to $\overline{AB}$.

Challenge Identify the points P and Q when ΔABC is equilateral.

1-9 Given that $ABCD$ is a square, $\overline{CF}$ bisects $\angle ACD$, and $\overline{BPQ}$ is perpendicular to CF (Fig. 1-9 prove $DQ = 2PE$.

1-9



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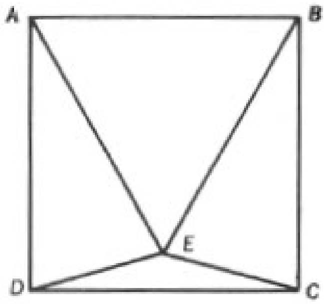


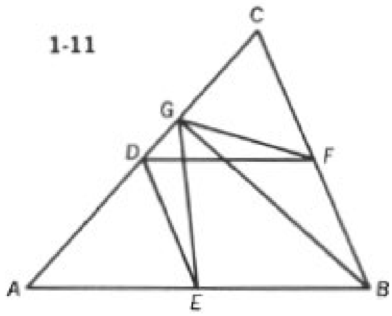
1-10 Given square *ABCD* with $m\angle EDC = m\angle ECD = 15$, prove ΔABC is equilateral (Fig. 1-10).

1-11 In any ΔABC, *D*, *E*, and *F* are midpoints of the sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$, respectively (Fig. 1-11 $\overline{BG}$ is an altitude of ΔABC. Prove that $\angle EGF \cong \angle EDF$.

Challenge 1 Investigate the case when ΔABC is equilateral.

Challenge 2 Investigate the case when $AC = CB$.

1-11



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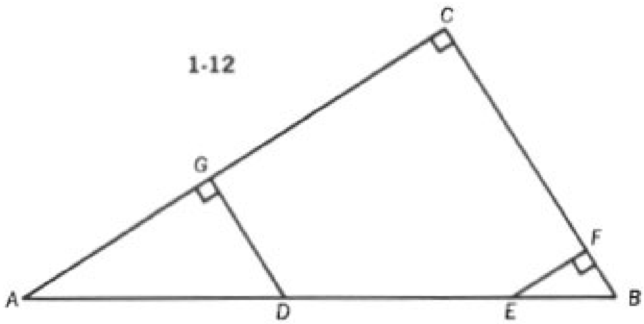


1-12 In right ΔABC, with right angle at *C*, $BD = BC$, $AE = AC$, $\overline{EF} \perp \overline{BC}$, and $\overline{DG} \perp \overline{AC}$ (Fig. 1-12 Prove that $DE = EF + DG$.

1-13 Prove that the sum of the measures of the perpendiculars from any point on a side of rectangle to the diagonals is constant.

Challenge If the point were on the extension of a side of the rectangle, would the result still hold?

1-14 The trisectors of the angles of a rectangle are drawn. For each pair of adjacent angles, tho trisectors that are closest to the enclosed side are extended until a point of intersection established. The line segments connecting those points of intersection form a quadrilatera Prove that the quadrilateral is a rhombus.

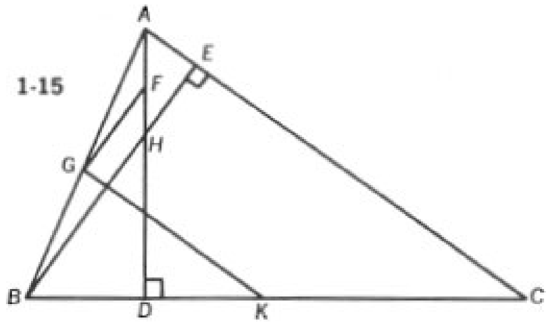
Challenge 1 What type of quadrilateral would be formed if the original rectangle were replaced l a square?

Challenge 2 What type of figure is obtained when the original figure is any parallelogram?

Challenge 3 What type of figure is obtained when the original figure is a rhombus?

1-15 In Fig. 1-15, $\overline{BE}$ and $\overline{AD}$ are altitudes of ΔABC. *F*, *G*, and *K* are midpoints of $\overline{AH}$, $\overline{AB}$, and $\overline{B}$

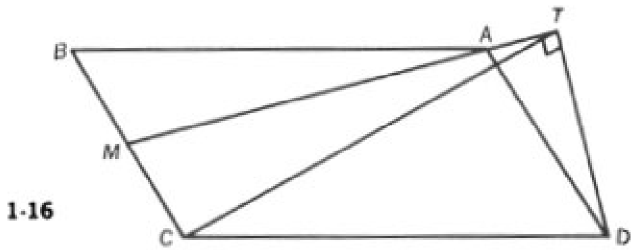
respectively. Prove that $\angle FGK$ is a right angle.



1-15

1-16 In parallelogram *ABCD*, *M* is the midpoint of $\overline{BC}$. $\overline{DT}$ is drawn from *D* perpendicular to $\overleftrightarrow{MA}$ in Fig. 1-16. Prove that $CT = CD$.

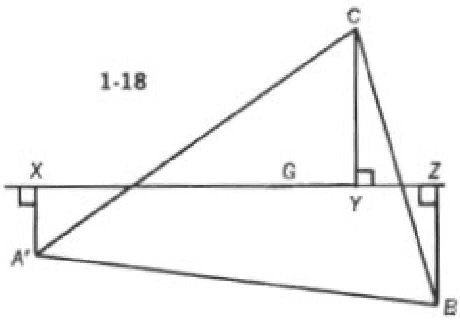
Challenge Make the necessary changes in the construction lines, and then prove the theorem for rectangle.



1-16

1-17 Prove that the line segment joining the midpoints of two opposite sides of any quadrilater bisects the line segment joining the midpoints of the diagonals.

1-18 In any ΔABC, $\overleftrightarrow{XYZ}$ is any line through the centroid *G* (Fig. 1-18). Perpendiculars are drav from each vertex of ΔABC to this line. Prove $CY = AX + BZ$.



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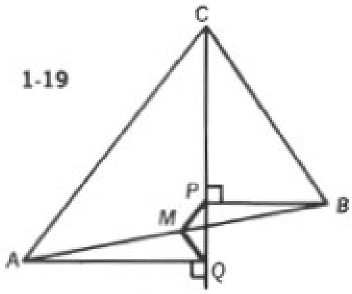


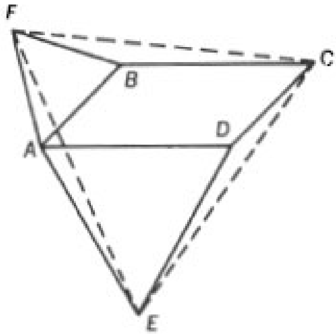
1-19

1-19 In any ΔABC, $\overleftrightarrow{CPQ}$ is any line through C, interior to ΔABC (Fig. 1-19). $\overline{BP}$ is perpendicular line CPQ, $\overline{AQ}$ is perpendicular to line CPQ, and *M* is the midpoint of $\overline{AB}$. Prove that *MP* *MQ*.

Challenge Show that the same result holds if the line through *C* is exterior to ΔABC.

1-20 In Fig. 1-20, *ABCD* is a parallelogram with equilateral triangles *ABF* and *ADE* drawn on sid $\overline{AB}$ and $\overline{AD}$, respectively. Prove that ΔFCE is equilateral.

1-20



1-21 If a square is drawn externally on each side of a parallelogram, prove that

(a) the quadrilateral determined by the centers of these squares is itself a square

(b) the diagonals of the newly formed square are concurrent with the diagonals of the origin parallelogram.

Challenge Consider other regular polygons drawn externally on the sides of a parallelogram. Stuc each of these situations!

2. Triangles in Proportion

As the title suggests, these problems deal primarily with similarity of triangles. Some interestin geometric proportions are investigated, and there is a geometric illustration of a harmonic mean.

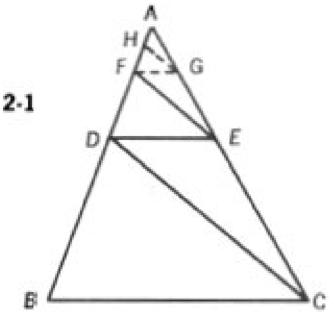
Do you remember manipulations with proportions such as: if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$? They a essential to solutions of many problems.

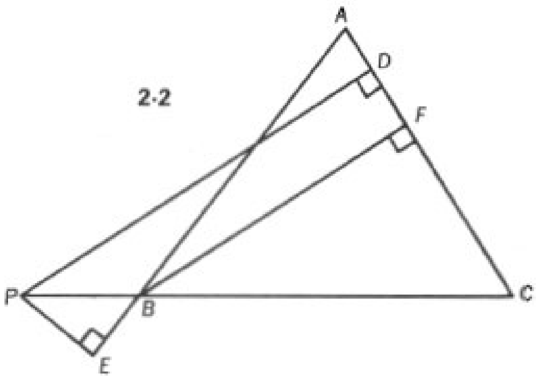
2-1 In ΔABC, $\overline{DE} \parallel \overline{BC}$, $\overline{FE} \parallel \overline{DC}$, $AF = 4$, and $FD = 6$ (Fig. 2-1). Find DB.

Challenge 1 Find DB if $AF = m_1$ and $FD = m_2$.

Challenge 2 $\overline{FG} \parallel \overline{DE}$, and $\overline{HG} \parallel \overline{FE}$. Find DB if $AH = 2$ and $HF = 4$.

Challenge 3 Find DB if $AH = m_1$ and $HF = m_2$.





2-2 In isosceles ΔABC ($AB=AC$), $\overline{CB}$ is extended through B to P (Fig. 2-2). A line from P, parall to altitude $\overline{BF}$, meets $\overline{AC}$ at D (where D is between A and F). From P, a perpendicular is draw to meet the extension of $\overline{AB}$ at E so that B is between E and A. Express BF in terms of PD a PE. Try solving this problem in two different ways.

Challenge Prove that $BF = PD + PE$ when $AB = AC$, P is between B and C, D is between C and and a perpendicular from P meets $\overline{AB}$ at E.

2-3 The measure of the longer base of a trapezoid is 97. The measure of the line segment joini the midpoints of the diagonals is 3. Find the measure of the shorter base.

Challenge Find a general solution applicable to any trapezoid.

2-4 In ΔABC, D is a pointy on side $\overline{BA}$ such that $BD{:}DA = 1{:}2$. E is a point on side $\overline{CB}$ so th $CE{:}EB = 1{:}4$. Segments $\overline{DC}$ and $\overline{AE}$ intersect at F. Express $CF{:}FD$ in terms of two positi relatively prime integers.

Challenge Show that if $BD{:}DA = m{:}n$ and $CE{:}EB = r{:}s$, then

$$\frac{CF}{FD} = \left(\frac{r}{s}\right)\left(\frac{m+n}{n}\right).$$

2-5 In ΔABC, $\overline{BE}$ is a median and O is the midpoint of $\overline{BE}$. Draw $\overline{AO}$ and extend it to meet $\overline{BC}$
~~D. Draw $\overline{CO}$ and extend it to meet $\overline{BA}$ at F. If $CO = 5$, $OF = 5$, and $AO = 12$, find the measu~~
of $\overline{OD}$.

Challenge Can you establish a relationship between OD and AO?

2-6 In parallelogram $ABCD$, points E and F are chosen on diagonal $\overline{AC}$ so that $AE = FC$. if $\overline{BE}$ extended to meet $\overline{AD}$ at H, and $\overline{BF}$ is extended to meet $\overline{DC}$ at G, prove that $\overline{HG}$ is parallel $\overline{AC}$.

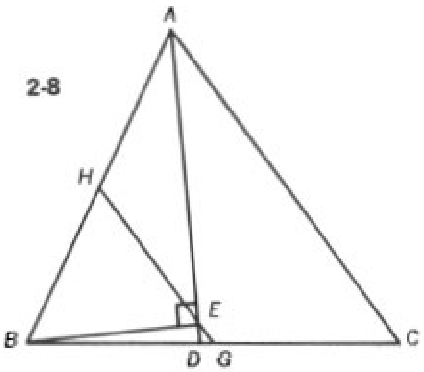
Challenge Prove the theorem if E and F are on $\overleftrightarrow{AC}$, exterior to the parallelogram.

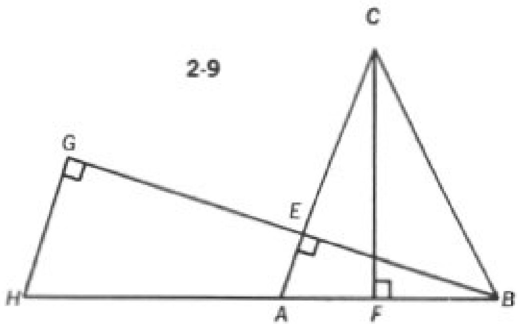
2-7 $\overline{AM}$ is the median to side $\overline{BC}$ of ΔABC, and P is any point on $\overline{AM}$. $\overline{BP}$ extended meets $\overline{AC}$ at and $\overline{CP}$ extended meets $\overline{AB}$ at D. Prove that $\overline{DE}$ is parallel to $\overline{BC}$.

Challenge Show that the result holds if P is on $\overleftrightarrow{AM}$, exterior to ΔABC.

2-8 In ΔABC, the bisector of $\angle A$ intersects $\overline{BC}$ at D (Fig. 2-8). A perpendicular to $\overline{AD}$ from intersects $\overline{AD}$ at E. A line segment through E and parallel to $\overline{AC}$ intersects $\overline{BC}$ at G, and $\overline{AB}$ H. If $AB = 26$, $BC = 28$, $AC = 30$, find the measure of $\overline{DG}$.

Challenge Prove the result for $\overleftrightarrow{CF} \perp \overleftrightarrow{AD}$ where F is on $\overleftrightarrow{AD}$ exterior to ΔABC.





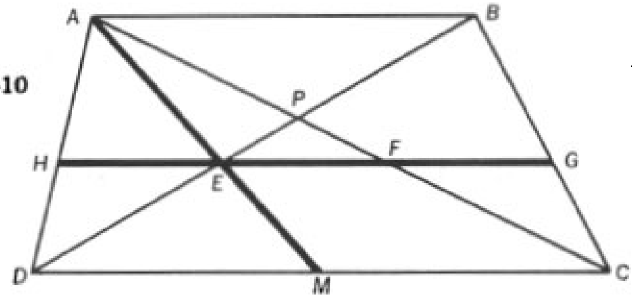
2-9 In ΔABC, altitude $\overline{BE}$ is extended to G so that EG = the measure of altitude $\overline{CF}$. A line throug G and parallel to $\overline{AC}$ meets $\overleftrightarrow{BA}$ at H, as in Fig. 2-9. Prove that $AH = AC$.

Challenge 1 Show that the result holds when $\angle A$ is a right angle.

Challenge 2 Prove the theorem for the case where the measure of altitude $\overline{BE}$ is greater than th measure of altitude $\overline{CF}$, and G is on $\overline{BE}$ (between B and E) so that EG = CF.

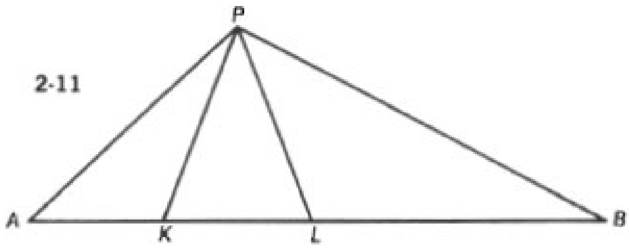
2-10 In trapezoid $ABCD$ ($\overline{AB} \parallel \overline{DC}$), with diagonals $\overline{AC}$ and $\overline{DB}$ intersecting at P, $\overline{AM}$, a median ΔADC, intersects $\overline{BD}$ at E (Fig. 2-10). Through E, a line is drawn parallel to $\overline{DC}$ cutting $\overline{A}$ $\overline{AC}$, and $\overline{BC}$ at points H, F, and G, respectively. Prove that $HE = EF = FG$.

2-10



2-11 A line segment $\overline{AB}$ is divided by points K and L in such a way that $(AL)^2 = (AK)(AB)$ (Fig. 11). A line segment $\overline{AP}$ is drawn congruent to $\overline{AL}$. Prove that $\overline{PL}$ bisects $\angle KPB$.

2-11

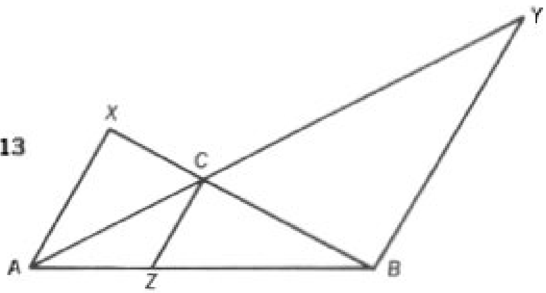


Challenge Investigate the situation when $\angle APB$ is a right angle.

2-12 P is any point on altitude $\overline{CD}$ of ΔABC. $\overline{AP}$ and $\overline{BP}$ meet sides $\overline{CB}$ and $\overline{CA}$ at points Q and respectively. Prove that $\angle QDC \cong \angle RDC$.

2-13 In ΔABC, Z is any point on base $\overline{AB}$ (Fig. 2-13). $\overline{CZ}$ is drawn. A line is drawn through parallel to $\overline{CZ}$ meeting $\overleftrightarrow{BC}$ at X. A line is drawn through B parallel to $\overline{CZ}$ meeting $\overleftrightarrow{AC}$ *at* Prove that $\frac{1}{AX} + \frac{1}{BY} = \frac{1}{CZ}$.

2-13



Challenge Two telephone cable poles, 40 feet and 60 feet high, respectively, are placed near eac other. As partial support, a line runs from the top of each pole to the bottom of the othe How high above the ground is the point of intersection of the two support lines?

2-14 In ΔABC, $m\angle A = 120$. Express the measure of the internal bisector of $\angle A$ in terms of the tw adjacent sides.

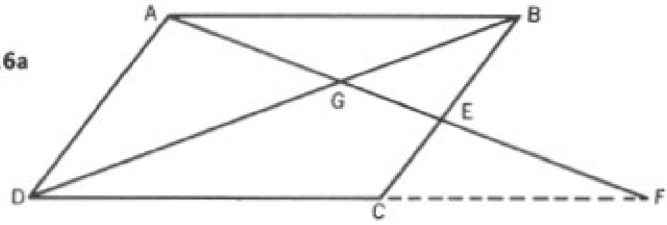
Challenge Prove the converse of the theorem established above.

2-15 Prove that the measure of the segment passing through the point of intersection of th diagonals of a trapezoid and parallel to the bases with its endpoints on the legs, is th harmonic mean between the measures of the parallel sides. The harmonic mean of tw numbers is defined as the reciprocal of the average of the reciprocals of two numbers. Th harmonic mean between a and b is equal to

$$\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} = \frac{2ab}{a+b}.$$

2-16 In ▱$ABCD$, E is on $\overline{BC}$ (Fig. 2-16a). $\overline{AE}$ cuts diagonal $\overline{BD}$ at G and $\overleftrightarrow{DC}$ at F. If $AG = 6$ and GE 4, find EF.

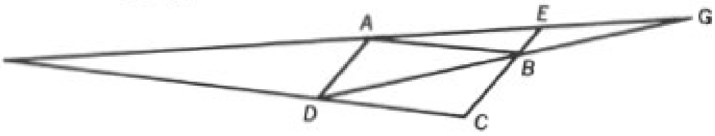
2-16a



Challenge 1 Show that AG is one-half the harmonic mean between AF and AE.

Challenge 2 Prove the theorem when E is on the extension of $\overline{CB}$ through B (Fig. 2-16b).

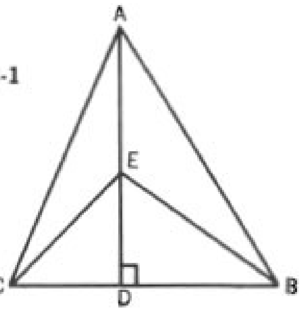
2-16b



You will find two kinds of problems in this section concerning the key result of Euclidean geometry, the theorem of Pythagoras. Some problems involve direct applications of the theorem. Others make use of results that depend on the theorem, such as the relationship between the sides of an isosceles right triangle or a 30–60–90 triangle.

3-1 In any ΔABC, E is any point on altitude $\overline{AD}$ (Fig. 3-1). Prove that $(AC)^2 - (CE)^2 = (AB)^2 - (EB)^2$.

3-1



Challenge 1 Show that the result holds if E is on the extension of $\overline{AD}$ through D.

Challenge 2 What change in the theorem results if E is on the extension of $\overline{AD}$ through A?

3-2 In ΔABC, median $\overline{AD}$ is perpendicular to median $\overline{BE}$. Find AB *if* $BC = 6$ and $AC = 8$.

Challenge 1 Express AB in general terms for $BC = a$, and $AC = b$.

Challenge 2 Find the ratio of AB to the measure of its median.

3-3 On hypotenuse $\overline{AB}$ of right ΔABC, draw square $ABLH$ externally. If $AC = 6$ and $BC = 8$, find CH.

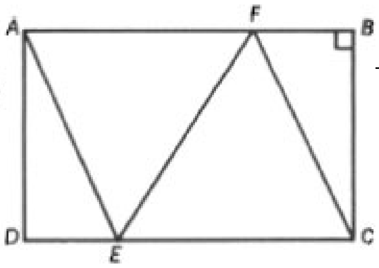
Challenge 1 Find the area of quadrilateral $HLBC$.

Challenge 2 Solve the problem if square $ABLH$ overlaps ΔABC.

3-4 The measures of the sides of a right triangle are 60, 80, and 100. Find the measure of a line segment, drawn from the vertex of the right angle to the hypotenuse, that divides the triangle into two triangles of equal perimeters.

3-5 On sides $\overline{AB}$ and $\overline{DC}$ of rectangle $ABCD$, points F and E are chosen so that $AFCE$ is a rhombus (Fig. 3-5). If $AB = 16$ and $BC = 12$, find EF.

3-5



Challenge If $AB = a$ and $BC = b$, what general expression will give the measure of $\overline{EF}$?

3-6 A man walks one mile east, then one mile northeast, then another mile east. Find the distanc in miles, between the man's initial and final positions.

Challenge How much shorter (or longer is the distance if the course is one mile east, one mi north, then one mile east?

3-7 If the measures of two sides and the included angle of a triangle are 7, $\sqrt{50}$, and 13 respectively, find the measure of the segment joining the midpoints of the two given sides.

Challenge 1 Show that when $m\angle A = 135$,

$$EF = \frac{1}{2}\sqrt{b^2 + c^2 + bc\sqrt{2}},$$

where E and F are midpoints of sides $\overline{AC}$ and $\overline{AB}$, respectively, of ΔABC.

NOTE: a, b, and c are the lengths of the sides opposite $\angle A$, $\angle B$, and $\angle C$ of ΔABC.

Challenge 2 Show that when $m\angle A = 120$,

$$EF = \frac{1}{2}\sqrt{b^2 + c^2 + bc\sqrt{1}}.$$

Challenge 3 Show that when $m\angle A = 150$,

$$EF = \frac{1}{2}\sqrt{b^2 + c^2 + bc\sqrt{3}}.$$

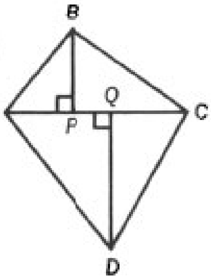
Challenge 4 On the basis of these results, predict the values of EF for $m\angle A = 30$, 45, 60, and 90.

3-8 Hypotenuse $\overline{AB}$ of right ΔABC is divided into four congruent segments by points G, E, and in the order A, G, E, H, B. If $AB = 20$, find the sum of the squares of the measures of the lin segments from C to G, E, and H.

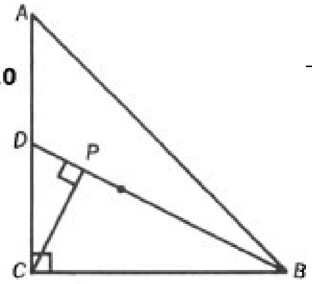
Challenge Express the result in general terms when $AB = c$.

3-9 In quadrilateral $ABCD$, $AB = 9$, $BC = 12$, $CD = 13$, $DA = 14$, and diagonal $AC = 15$ (Fig. 3-9 Perpendiculars are drawn from B and D to $\overline{AC}$, meeting $\overline{AC}$ at points P and Q, respectivel Find PQ.

3-9



3-10



3-10 In ΔABC, angle C is a right angle (Fig. 3-10). $AC = BC = 1$, and D is the midpoint of $\overline{AC}$. $\overline{B}$ is drawn, and a line perpendicular to $\overline{BD}$ at P is drawn from C Find the distance from P to th intersection of the medians of ΔABC.

Challenge Show that $PG = \frac{c\sqrt{10}}{30}$, when G is the centroid, and c is the length of the hypotenuse.

3-11 A right triangle contains a 60° angle. If the measure of the hypotenuse is 4, find the distanc from the point of intersection of the 2 legs of the triangle to the point of intersection of th angle bisectors.

3-12 From point P inside ΔABC, perpendiculars are drawn to the sides meeting $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, points D, E, and F, respectively. If $BD = 8$, $DC = 14$, $CE = 13$, $AF = 12$, and $FB = 6$, find A Derive a general theorem, and then make use of it to solve this problem.

3-13 For ΔABC with medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, let $m = AD + BE + CF$, and let $s = AB + BC + C$ Prove that $\frac{3}{2}s > m > \frac{3}{4}s$.

3-14 Prove that $\frac{3}{4}(a^2 + b^2 + c^2) = m_a{}^2 + m_b{}^2 + m_c{}^2$. ($m_c$ means the measure of the median draw to side c.)

Challenge 1 Verify this relation for an equilateral triangle.

Challenge 2 The sum of the squares of the measures of the sides of a triangle is 120. If two of th medians measure 4 and 5, respectively, how long is the third median?

Challenge 3 If $\overline{AE}$ and $\overline{BF}$ are medians drawn to the legs of right ΔABC, find the numeral value o $\frac{(AE)^2 + (BF)^2}{(AB)^2}$.

4. Circles Revisited

Circles are the order of the day in this section. There are problems dealing with arc and ang measurement; others deal with lengths of chords, secants, tangents, and radii; and some problen involve both.

Particular attention should be given to Problems 4-33 thru 4-40, which concern cycl quadrilaterals (quadrilaterals that may be inscribed in a circle). This often neglected subject h interesting applications. If you are not familiar with it, you might look at the theorems that are liste in Appendix I.

4-1 Two tangents from an external point *P* are drawn to a circle, meeting the circle at points *A* ar *B*. A third tangent meets the circle at *T*, and tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at points *Q* and *R*, respectivel Find the perimeter *p* of ΔPQR.

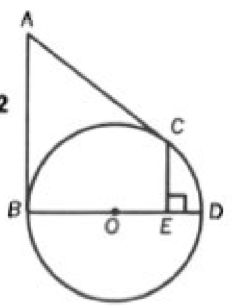
4-2 $\overline{AB}$ and $\overline{AC}$ are tangent to circle *O* at *B* and C, respectively, and $\overline{CE}$ is perpendicular diameter $\overline{BD}$ (Fig. 4-2). Prove $(BE)(BO) = (AB)(CE)$.

Challenge 1 Find the value of *AB* when *E* coincides with *O*.

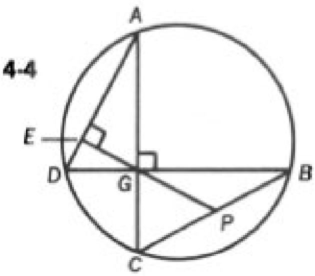
Challenge 2 Show that the theorem is true when *E* is between *B* and *O*.

Challenge 3 Show that $\frac{AB}{\sqrt{BE}} = \frac{BO}{\sqrt{ED}}$.

4-2



4-4



4-3 From an external point *P*, tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn to a circle. From a point *Q* on tl major (or minor) arc $\overset{\frown}{AB}$, perpendiculars are drawn to $\overline{AB}$, $\overrightarrow{PA}$, and $\overrightarrow{PB}$. Prove that tl perpendicular to $\overline{AB}$ is the mean proportional between the other two perpendiculars.

Challenge Show that the theorem is true when the tangents are parallel.

4-4 Chords $\overline{AC}$ and $\overline{DB}$ are perpendicular to each other and intersect at point *G* (Fig. 4-4). ΔAGD the altitude from *G* meets $\overline{AD}$ at *E*, and when extended meets $\overline{BC}$ at *P*. Prove that *BP* *PC*.

Challenge One converse of this theorem is as follows. Chords $\overline{AC}$ and $\overline{DB}$ intersect at *G*. In ΔAG the altitude from *G* meets $\overline{AD}$ at *E*, and when extended meets $\overline{BC}$ at *P* so that $BP = P$ Prove that $\overline{AC} \perp \overline{BD}$.

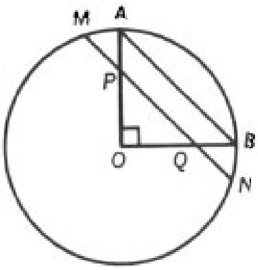
4-5 Square *ABCD* is inscribed in a circle. Point *E* is on the circle. If $AB = 8$, find the value of

$(AE)^2 + (BE)^2 + (CE)^2 + (DE)^2$.

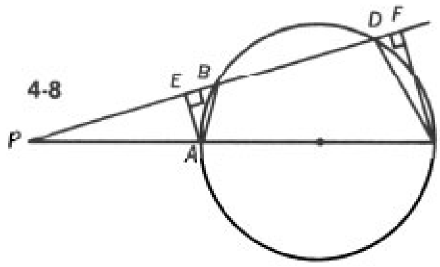
Challenge Prove that for *ABCD*, a non-square rectangle, $(AE)^2 + (BE)^2 + (CE)^2 + (DE)^2 = 2d$ where *d* is the measure of the length of a diagonal of the rectangle.

4-6 Radius $\overline{AO}$ is perpendicular to radius $\overline{OB}$, $\overline{MN}$ is parallel to $\overline{AB}$ meeting $\overline{AO}$ at *P* and $\overline{OB}$ at and the circle at *M* and *N* (Fig. 4-6). If MP = $\sqrt{56}$, and *PN* = 12, find the measure of the radi of the circle.

4-6



4-8



4-7 Chord $\overline{CD}$ is drawn so that its midpoint is 3 inches from the center of a circle with a radius 6 inches. From *A*, the midpoint of minor arc $\widehat{CD}$, any chord $\overline{AB}$ is drawn intersecting $\overline{CD}$ in Let *v* be the range of values of (*AB*)(*AM*), as chord $\overline{AB}$ is made to rotate in the circle about t fixed point *A*. Find *v*.

4-8 A circle with diameter $\overline{AC}$ is intersected by a secant at points *B* and *D*. The secant and t diameter intersect at point *P* outside the circle, as shown in Fig. 4-8. Perpendiculars $\overline{AE}$ a $\overline{CF}$ are drawn from the extremities of the diameter to the secant. If *EB* = 2, and *BD* = 6, fi *DF*.

Challenge Does *DF* = *EB*? Prove it!

4-9 A diameter $\overline{CD}$ of a circle is extended through *D* to external point *P*. The measure of seca $\overline{CP}$ is 77. From *P*, another secant is drawn intersecting the circle first at *A*, then at *B*. T measure of secant $\overline{PB}$ is 33. The diameter of the circle measures 74. Find the measure of t angle formed by the secants.

Challenge Find the measure of the shorter secant if the measure of the angle between the secants 45.

4-10 In Δ*ABC*, in which *AB* = 12, *BC* = 18, and *AC* = 25, a semicircle is drawn so that its diamet lies on $\overline{AC}$, and so that it is tangent to $\overline{AB}$ and $\overline{BC}$. If *O* is the center of the circle, find t measure of $\overline{AO}$.

Challenge Find the diameter of the semicircle.

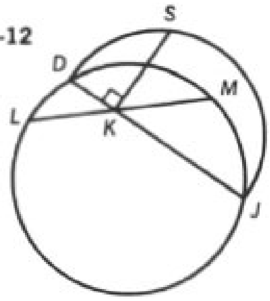
4-11 Two parallel tangents to circle *O* meet the circle at points *M* and *N*. A third tangent to circle at point *P*, meets the other two tangents at points *K* and *L*. Prove that a circle, whose diamet is $\overline{KL}$, passes through *O*, the center of the original circle.

Challenge Prove that for different positions of point *P*, on $\widehat{MN}$, a family of circles is obtain tangent to each other at *O*.

4-12 $\overline{LM}$ is a chord of a circle, and is bisected at K (Fig. 4-12). $\overline{DKJ}$ is another chord. A semicirc
is drawn with diameter $\overline{DJ}$. $\overline{KS}$, perpendicular to $\overline{DJ}$, meets this semicircle at S. Prove KS
KL.

Challenge Show that if $\overline{DKJ}$ is a diameter of the first circle, or if $\overline{DKJ}$ coincides with $\overline{LM}$, th
theorem is trivial.

4-12



4-13 ΔABC is inscribed in a circle with diameter $\overline{AD}$. A tangent to the circle at D cuts $\overline{AB}$ extende
at E and $\overline{AC}$ extended at F. If $AB = 4$, $AC = 6$, and $BE = 8$, find CF.

Challenge 1 Find $m\angle DAF$.

Challenge 2 Find BC.

4-14 Altitude $\overline{AD}$ of equilateral ΔABC is a diameter of circle O. If the circle intersects $\overline{AB}$ and $\overline{A}$
at E and F, respectively, find the ratio of $EF : BC$.

Challenge Find the ratio of EB:BD.

4-15 Two circles intersect in A and B, and the measure of the common chord $\overline{AB}$ is 10. The li
joining the centers cuts the circles in P and Q. If $PQ = 3$ and the measure of the radius of o
circle is 13, find the radius of the other circle.

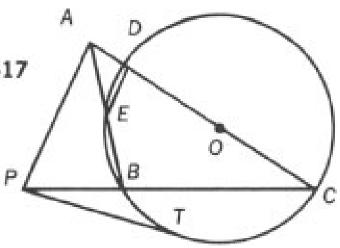
Challenge Find the second radius if $PQ = 2$.

4-16 $ABCD$ is a quadrilateral inscribed in a circle. Diagonal $\overline{BD}$ bisects $\overline{AC}$. If $AB = 10$, $AD = 1$
and $DC = 11$, find BC.

Challenge Solve the problem when diagonal $\overline{BD}$ divides $\overline{AC}$ into two segments, one of which
twice as long as the other.

4-17 A is a point exterior to circle O. $\overline{PT}$ is drawn tangent to the circle so that $PT = PA$. As shown
Fig. 4-17, C is any point on circle O, and $\overline{AC}$ and $\overline{PC}$ intersect the circle at points D and
respectively. $\overline{AB}$ intersects the circle at E. Prove that $\overline{DE}$ is parallel to $\overline{AP}$.

4-17



Challenge 1 Prove the theorem for A interior to circle O.

Challenge 2 Explain the situation when A is on circle O.

4-18 $\overline{PA}$ and $\overline{PB}$ are tangents to a circle, and $\overline{PCD}$ is a secant. Chords $\overline{AC}$, $\overline{BC}$, $\overline{BD}$, and $\overline{DA}$ a drawn. If $AC = 9$, $AD = 12$, and $BD = 10$, find BC.

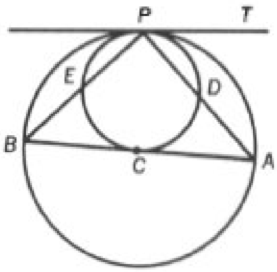
Challenge If in addition to the information given above, $PA = 15$ and $PC = 9$, find AB.

4-19 The altitudes of ΔABC meet at O. $\overline{BC}$, the base of the triangle, has a measure of 16. Th circumcircle of ΔABC has a diameter with a measure of 20. Find AO.

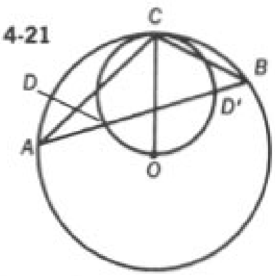
4-20 Two circles are tangent internally at P, and a chord, $\overline{AB}$, of the larger circle is tangent to th smaller circle at C (Fig. 4-20). $\overline{PB}$ and $\overline{PA}$ cut the smaller circle at E and D, respectively. If A $= 15$, while $PE = 2$ and $PD = 3$, find AC.

Challenge Express AC in terms of AB, PE, and PD.

4-20



4-21



4-21 A circle, center O, is circumscribed about ΔABC, a triangle in which $\angle C$ is obtuse (Fig. 4-21 With $\overline{OC}$ as diameter, a circle is drawn intersecting $\overline{AB}$ in D and D'. If $AD = 3$, and $DB =$ find CD.

Challenge 1 Show that the theorem is or is not true if $m\angle C = 90$.

Challenge 2 Investigate the case for $m\angle C < 90$.

4-22 In circle O, perpendicular chords $\overline{AB}$ and $\overline{CD}$ intersect at E so that $AE = 2$, $EB = 12$, and CE 4. Find the measure of the radius of circle O.

Challenge Find the shortest distance from E to the circle.

4-23 Prove that the sum of the measure of the squares of the segments made by two perpendicul chords is equal to the square of the measure of the diameter of the given circle.

Challenge Prove the theorem for two perpendicular chords meeting outside the circle.

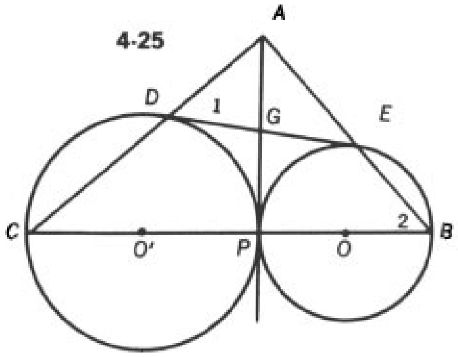
4-24 Two equal circles are tangent externally at T. Chord $\overline{TM}$ in circle O is perpendicular to cho $\overline{TN}$ in circle Q. Prove that $\overline{MN} \parallel \overline{OQ}$ and $MN = OQ$.

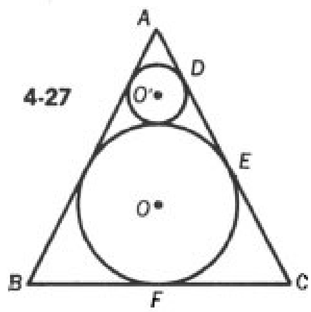
Challenge Show that $MN = \sqrt{2(R^2 + r^2)}$ if the circles are unequal, where R and r are the radii the two circles.

4-25 From point A on the common internal tangent of tangent circles O and O', secants $\overline{AEB}$ ar $\overline{ADC}$ are drawn, respectively (Fig. 4-25). If $\overline{DC}$ is the common external tangent, and points and B are collinear with the centers of the circles, prove

(a) $m\angle 1 = m\angle 2$, and

(b) $\angle A$ is a right angle.





Challenge 1 Prove or disprove that if $\overline{BC}$ does not pass through the centers of the circles, tl designated pairs of angles are not equal and $\angle A$ is not a right angle.

Challenge 2 Prove that DE is the mean proportional between the diameters of circles O and O'.

4-26 Two equal intersecting circles O and O' have a common chord $\overline{RS}$. From any point P on $\overline{RS}$ ray is drawn perpendicular to $\overline{RS}$ cutting circles O and O' at A and B, respectively. Prove th $\overline{AB}$ is parallel to the line of centers, $\overleftrightarrow{OO'}$, and that $AB = OO'$.

4-27 A circle is inscribed in a triangle whose sides are 10, 10, and 12 units in measure (Fig. 4-27 A second, smaller circle is inscribed tangent to the first circle and to the equal sides of tl triangle. Find the measure of the radius of the second circle.

Challenge 1 Solve the problem in general terms if $AC = a$, $BC = 2b$.

Challenge 2 Inscribe a third, smaller circle tangent to the second circle and to the equal sides, ar find its radius by inspection.

Challenge 3 Extend the legs of the triangle through B and C, and draw a circle tangent to tl original circle and to the extensions of the legs. What is its radius?

4-28 A circle with radius 3 is inscribed in a square. Find the radius of the circle that is inscribe between two sides of the square and the original circle.

Challenge Show that the area of the small circle is approximately 3% of the area of the large circl

4-29 $\overline{AB}$ is a diameter of circle O, as shown in Fig. 4-29. Two circles are drawn with $\overline{AO}$ and $\overline{OB}$ diameters. In the region between the circumferences, a circle D is inscribed, tangent to tl three previous circles. If the measure of the radius of circle D is 8, find AB.

Challenge Prove that the area of the shaded region equals the area of circle E.