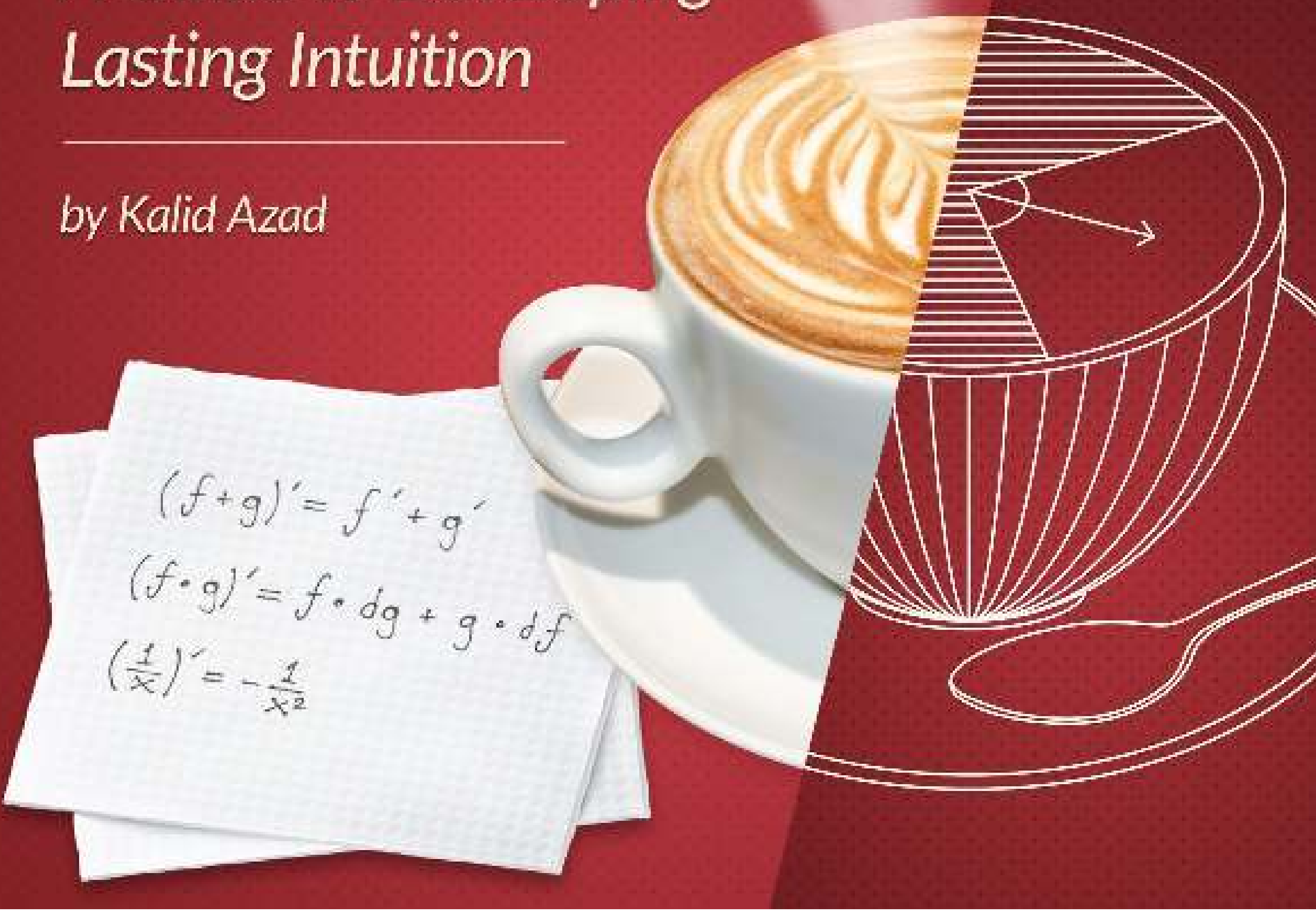
Calculus

Better Explained

A Guide to Developing Lasting Intuition

by Kalid Azad



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CHAPTER 1

1 Minute Calculus: X-Ray and Time-Lapse Vision

We usually take shapes, formulas, and situations at face value. Calculus gives us two superpowers to dig deeper:

- **X-Ray Vision:** You see the hidden pieces inside a pattern. You don't just see the tree, you know it's made of rings, with another growing as we speak.

- **Time-Lapse Vision:** You see the future path of an object laid out before you (cool, right?). "Hey, there's the moon. For the next few days it'll be white, but on the sixth it'll be low in the sky, in a color I like. I'll take a photo then."

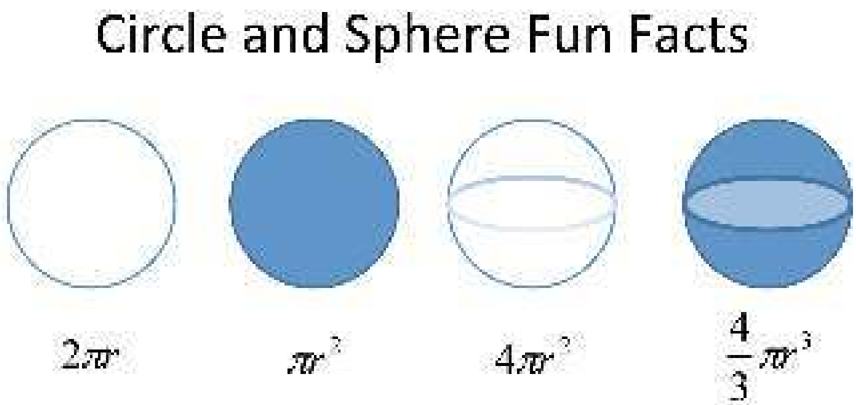
So why is Calculus useful? Well, just imagine having X-Ray or Time-Lapse vision to use at will. That object over there, how was it put together? What will happen to it?

(Strangely, my letters to Marvel about Calculus-man have been ignored to date.)

1.1 Calculus In 10 Minutes: See Patterns Step-By-Step

What do X-Ray and Time-Lapse vision have in common? They examine patterns step-by-step. An X-Ray shows the individual slices inside, and a time-lapse puts each future state next to the other.

This seems pretty abstract. Let's look at a few famous patterns:

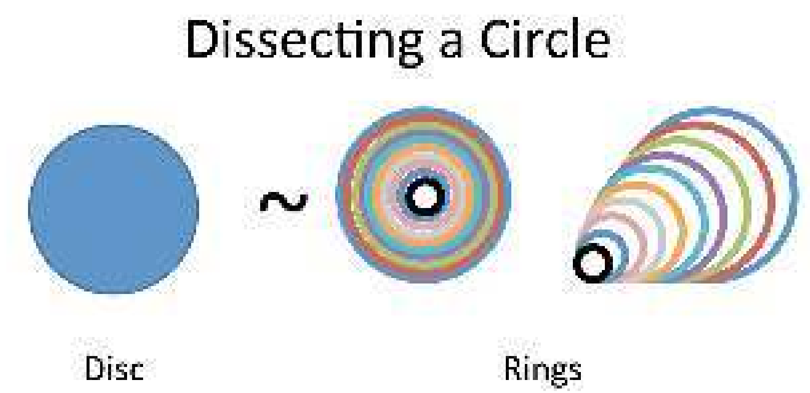


We have a vague feeling these formulas are connected, right?

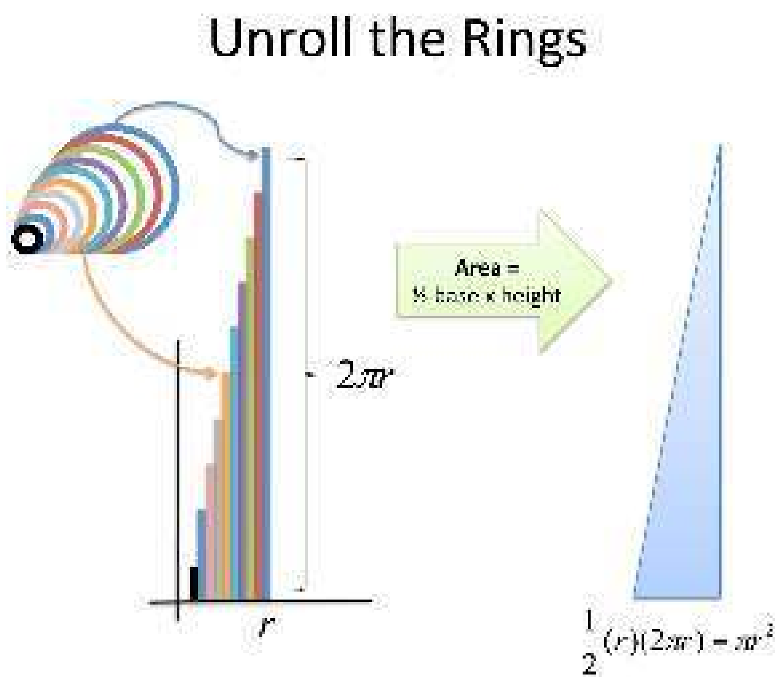
Let's turn our X-Ray vision and see where this leads. Suppose we know the equation for circumference ($2\pi r$) and want to figure out the equation for area. What can we do?

This is a tough question. Squares are easy to measure, but what can we do with an ever-curving shape?

Calculus to the rescue. Let's use our X-Ray vision to realize a disc is really just a bunch of rings put together. Similar to a tree trunk, here's a "step-by-step" view of a filled-in circle:



Why does this viewpoint help? Well, let's unroll those curled-up rings so they're easier to measure:



Whoa! We have a bunch of straightened rings that form a triangle, which is much easier to measure (Wikipedia has an animation).

The height of the largest ring is the full circumference ($2\pi r$), and each ring gets smaller. The height of each ring depends on its original distance from the center; the ring 3 inches from the center has a height of $2\pi \cdot 3$ inches. The smallest ring is a pinpoint, more or less, without any height at all.

And because triangles are easier to measure than circles, finding the area isn't too much trouble. The area of the "ring triangle" = $\frac{1}{2} base \cdot height = \frac{1}{2} r(2\pi r) = \pi r^2$, which is the formula for a circle's area!

Our X-Ray vision revealed a simple, easy-to-measure structure within a curvy shape. We realized a circle and a set of glued-together rings were really the same. From another perspective, a filled-in disc is the "time lapse" of a ring that got larger and larger.

Remember learning arithmetic? We learned a few things to do with numbers (and equations): add/subtract, multiply/divide, and use exponents/roots. Not a bad start.

Calculus gives us two new options: split apart and glue together. A key epiphany of calculus is that our existing patterns can be seen as a bunch of glued-together pieces. It's like staring at a building and knowing it was made brick-by-brick.

1.2 So... What Can I Do With Calculus?

It depends. What can you do with arithmetic?

Technically, we don't need numbers. Our caveman ancestors did fine (well, for some version of "fine"). But having an idea of quantity makes the world a lot easier. You don't have a "big" and "small" pile of rocks: you have an exact count. You don't need to describe your mood as "good" or "great" – you can explain your mood from 1 to 10.

Arithmetic provides universal metaphors that go beyond computation, and once seen, they're difficult to give up.

Calculus is similar: it's a step-by-step view of the world. Do we need X-Ray and Time Lapse vision all the time? Nope. But they're nice perspectives to "turn on" when we face a puzzling situation. What steps got us here? What's the next thing that will happen? Where does our future path lead?

There are specific rules to calculus, just like there are rules to arithmetic. And they're quite useful when you're cranking through an equation. But don't forget there are general notions of "quantity" and "step-by-step thinking" that we can bring everywhere.

Our primary goal is to *feel* what a calculus perspective is like (*What Would Archimedes Do?*). Over time, we'll work up to using the specific rules ourselves.

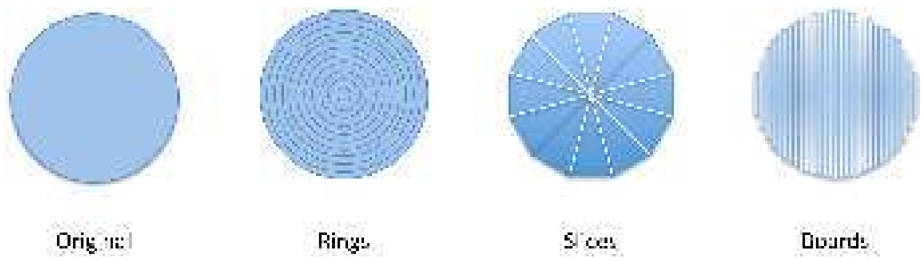
CHAPTER 2

Practice X-Ray and Time-Lapse Vision

Calculus trains us to use X-Ray and Time-Lapse vision, such as re-arranging a circle into a “ring triangle” (diagram). This makes finding the area... well, if not exactly *easy*, much more manageable.

But we were a little presumptuous. Must *every* circle in the universe be made from rings?

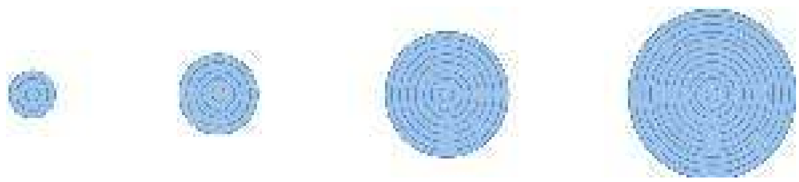
Heck no! We’re more creative than that. Here’s a few more options for our X-Ray:



Now we’re talking. We can imagine a circle as a set of rings, pizza slices, or vertical boards. Each underlying “blueprint” is a different step-by-step strategy in action.

Imagine each strategy unfolding over time, using your time-lapse vision. Have any ideas about what each approach is good for?

Ring-by-ring Analysis



Rings are the old standby. What’s neat about a ring-by-ring progression?

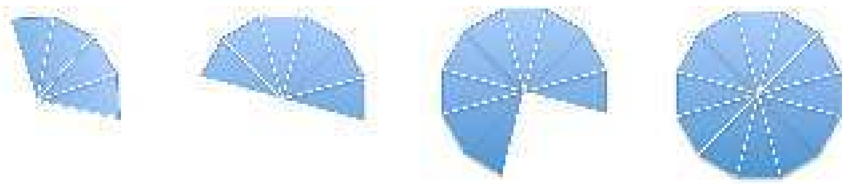
- Each intermediate stage is an entire “mini circle” on its own. i.e., when we’re halfway done, we still have a circle, just one with half the regular radius.
- Each step is an increasing amount of work. Just imagine plowing a circular field and spreading the work over several days. On the first day, you start at the center and don’t even move. The next, you make take the tightest turn you can. Then you start doing laps, larger and larger, until you are circling the entire yard on the last day. (Note: The change between each ring is the same; maybe it’s 1 extra minute each time you make the ring larger).
- The work is reasonably predictable, which may help planning. If we know it’s an extra minute for each lap, then the 20th ring will take 20 minutes.
- Most of the work happens in the final laps. In the first 25% of the time-lapse, we’ve barely grown: we’re adding tiny rings. Near the end, we start to pick up steam by adding long slices, each nearly the final size.

Now let’s get practical: why is it that trees have a ring pattern inside?

It’s the first property: **a big tree must grow from a complete smaller tree**. With the ring-by-ring strategy, we’re always adding to a complete, fully-formed circle. We aren’t trying to grow the “left half” of the tree and then work on the right side.

In fact, many natural processes that grow (trees, bones, bubbles, etc.) take this inside-out approach.

Slice-by-slice Analysis



Now think about a slice-by-slice progression. What do you notice?

- We contribute the same amount with each step. Even better, the parts are identical. This may not matter for math, but in the real world (aka cutting a cake), we make the same action when cutting out each slice, which is convenient.
- Since the slices are symmetrical, we can use shortcuts like making cuts across the entire shape to speed up the process. These “assembly line” speedups work well for identical components.
- Progress is extremely easy to measure. If we have 10 slices, then at slice 6 we are exactly 60% done (by both area and circumference).

- We follow a sweeping circular path, never retracing our steps from an "angular" point of view. When carving out the rings, we went through the full 360-degrees on each step.

Time to think about the real world. What follows this slice-by-slice pattern, and why?

Well food, for one. Cake, pizza, pie: we want everyone to have an equal share. Slices are simple to cut, we get nice speedups (like cutting across the cake), and it's easy to see how much is remaining. (Imagine cutting circular rings from a pie and trying to estimate how much area is left.)

Now think about radar scanners: they sweep out in a circular path, "clearing" a slice of sky before moving to another angle. This strategy does leave a blind spot in the angle you haven't yet covered, a tradeoff you're hopefully aware of.

Contrast this to sonar used by a submarine or bat, which sends a sound "ring" propagating in every direction. That works best for close targets (covering every direction at once). The drawback is that unfocused propagation gets much weaker the further out you go, as the initial energy is spread out over a larger ring. We use megaphones and antennas to focus our signals into beams (thin slices) to get the max range for our energy.

Operationally, if we're building circular shape from a set of slices (like a paper fan), it helps to have every part be identical. Figure out the best way to make a single slice, then mass produce them. Even better: if one slice can collapse, the entire shape can fold up!

Board-by-board Analysis

Getting the hang of X-Rays and Time-lapses? Great. Look at the progression above, and spend a few seconds thinking of the pros and cons. Don't worry, I'll wait.

Ready? Ok. Here's a few of my observations:

- This is a very robotic pattern, moving left-to-right and never returning to a previous horizontal position.
- The contribution from each step starts small, gradually gets larger, maxes out in the middle, and begins shrinking again.
- Our progress is somewhat unpredictable. Sure, at the halfway mark we've finished half the circle, but the pattern rises and falls which makes it difficult to analyze. By contrast, the ring-by-ring pattern changed the same amount each time, always increasing. It was clear that the later

rings would add the most work. Here, it's the middle section which seems to be doing the heavy lifting.

Ok, time to figure out where this pattern shows up in the real world.

Decks and wooden structures, for one. When putting down wooden planks, we don't want to retrace our steps, or return to a previous position (especially if there are other steps involved, like painting). Just like a tree needs a fully-formed circle at each step, a deck insists upon components found at Home Depot (i.e., rectangular boards).

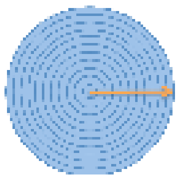
In fact, any process with a linear "pipeline" might use this approach: finish a section and move onto the next. Think about a printer that has to spray a pattern top-to-bottom as the paper is fed through (or these days, a 3d printer). It doesn't have the luxury of a ring-by-ring or a slice-by-slice approach. It will see a horizontal position only once, so it better make it count!

From a human motivation perspective, it may be convenient to start small, work your way up, then ease back down. A pizza-slice approach could be tolerable (identical progress every day), but rings could be demoralizing: every step requires more than the one before, without yielding.

Getting Organized

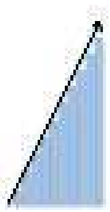
So far, we've been using natural descriptions to explain our thought processing. "Take a bunch of rings" or "Cut the circle into pizza slices". This conveys a general notion, but it's a bit like describing a song as "Dum-de-dum-dum" – you're pretty much the only one who knows what's happening. A little organization can make it perfectly clear what we mean.

The first thing we can do is keep track of how we're making our steps. I like to imagine a little arrow in the direction we move as we take slices:



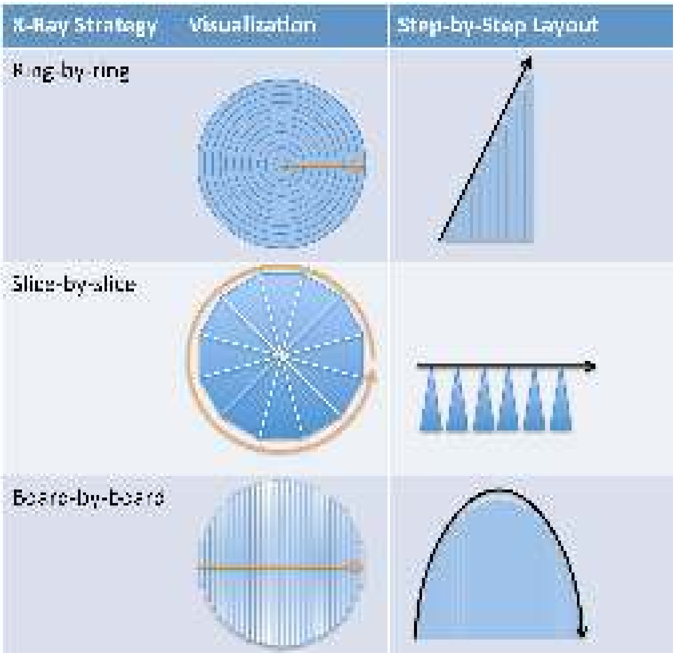
In my head, I'm moving along the yellow line, calling out the steps like Oprah giving away cars (*you* get a ring, *you* get a ring, *you* get a ring...). (Hey, it's my analogy, don't give me that look!)

The arrow is handy, but it's still tricky to see the exact progression of slices. Why don't we explicitly "line up" the changes? As we saw before, we can unroll the steps, put them side-by-side, and make them easier to compare:



The black arrow shows the trend. Pretty nice, right? We can tell, at a glance, that the slices are increasing, and by the same amount each time (since the trend line is straight).

Math fans and neurotics alike enjoy these organized layouts; there is something soothing about it, I suppose. And since you're here, we might as well organize the other patterns too:



Now it's much easier to compare each X-Ray strategy:

- With rings, steps increase steadily (upward sloping line)
- With slices, steps stay the same (flat line)
- With boards, steps get larger, peak, then get smaller (up and down; note, the curve looks elongated because the individual boards are lined up on the bottom)

The charts made our comparisons easier, wouldn't you say? Sure. But wait, isn't that trendline looking like a dreaded x-y graph?

Yep. Remember, a graph is a visual explanation that should *help* us. If it's confusing, it needs to be fixed.

Many classes present graphs, divorced from the phenomena that made them, and hope you see an invisible sequence of steps buried inside. It's a recipe for pain – just be explicit about what a graph represents!

Archimedes did fine without x-y graphs, finding the area of a circle using the "ring-to-triangle" method. In this primer we'll leave our level of graphing to what you see above (the details of graphs will be a nice follow-up, after our intuition is built).

So, are things starting to click a bit? Thinking better with X-Rays and Time-lapses?

PS. It may bother you that our steps create a "circle-like" shape, but not a *real, smooth* circle. We'll get to that :). But to be fair, it must also bother you that the square pixels on this screen make "letter-like" shapes, and not *real, smooth* letters. And somehow, the "letter-like pixels" convey the same meaning as the real thing!

Questions

1) Can you describe a grandma-friendly version of what you've learned?

2) Let's expand our thinking into the 3rd dimension. Can you of a few ways to build a sphere? (No formulas, just descriptions)

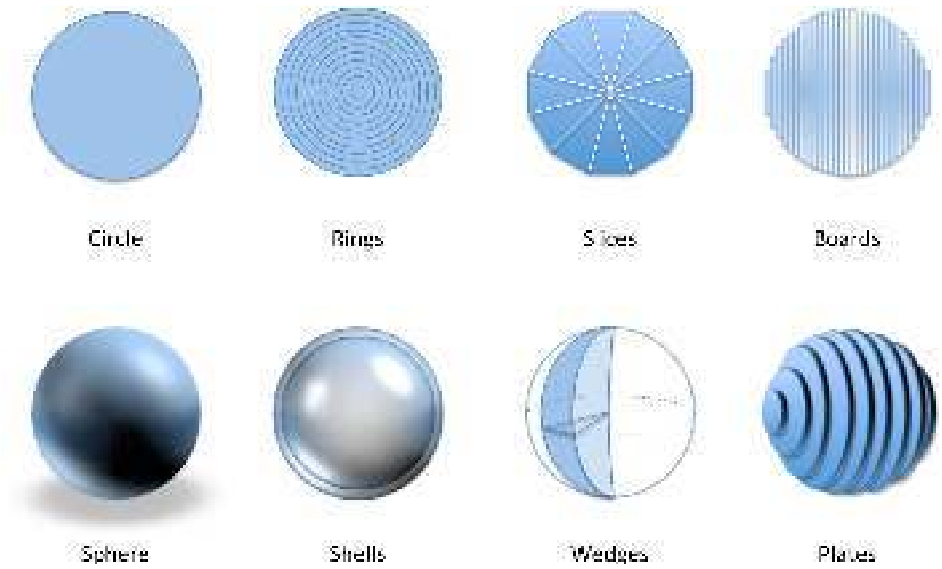
I'll share a few approaches with you in the next lesson. Happy math.

Chapter 3

Expanding Our Intuition

Hope you thought about the question from last time: how do we take our X-Ray strategies into the 3rd dimension?

Here's my take:



- Rings become *shells*, a thick candy coating on a delicious gobstopper. Each layer is slightly bigger than the one before.
- Slices become *wedges*, identical sections like slices of an orange.
- Boards become *plates*, thick discs which can be stacked together. (I sometimes daydream of opening a bed & breakfast that only serves spherical stacks of pancakes. *And they say math has no real-world uses!*)

The 3d steps can be seen as the 2d versions, swept out in various ways. For example, spin the individual rings (like a coin) to create a shell. Imagine slices pushed through a changing mold to make wedges. Lastly, imagine we spin the boards to make plates, like carving a wooden sphere with a lathe (video).

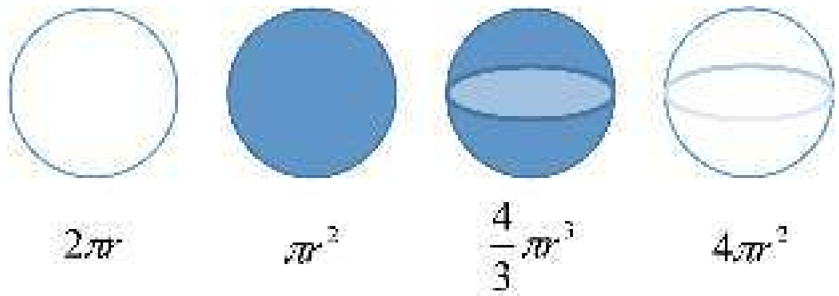
The tradeoffs are similar to the 2d versions:

- Organic processes grow in layers (pearls in an oyster)
- Fair divisions require wedges (cutting an apple for friends)
- The robotic plate approach seems easy to manufacture

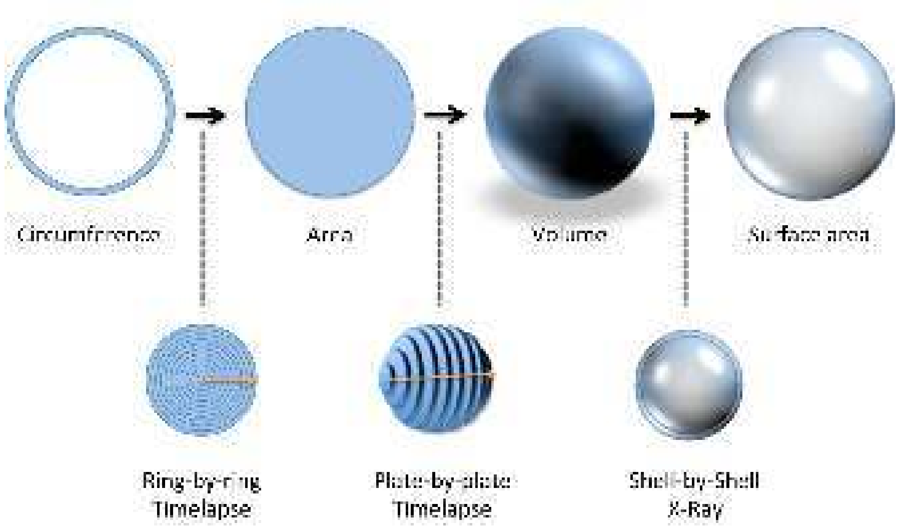
An orange is an interesting hybrid: from the outside, it appears to be made from shells, growing over time. And inside, it forms a symmetric internal structure – a better way to distribute seeds, right? We could analyze it both ways.

Exploring The 3d Perspective

In the first lesson we had the vague notion the circle/sphere formulas were related:



Well, now we have an idea how:



- **Circumference**: Start with a single ring
- **Area:** Make a filled-in disc with a ring-by-ring time lapse
- **Volume:** Make the circle into a plate, and do a plate-by-plate time lapse to build a sphere
- **Surface area:** X-Ray the sphere into a bunch of shells; the outer shell is the surface area

Wow! These descriptions are pretty detailed. We know, intuitively, how to morph shapes into alternate versions by thinking “time-lapse this” or “X-Ray that”. We can move backwards, from a sphere back to circumference, or try different strategies: maybe we want to split the circle into boards, not rings.

The Need For Math Notation

You might have noticed it’s getting harder to explain your ideas. We’re reaching for physical analogies (rings, boards, wedges) to explain our plans: “Ok, take that circular area, and try to make some discs out of it. Yeah, like that. Now line them up into the shape of a sphere...”.

I love diagrams and analogies, but should they be *required* to explain an idea? Probably not.

Take a look how numbers developed. At first, we used very literal symbols for counting: I, II, III, and so on. Eventually, we realized a symbol like V could take the place of IIIII, and even better, that every digit could have its own symbol (we do keep our metaphorical history with the number 1).

This math abstraction helped in a few ways:

- **It’s shorter.** Isn’t “2 + 3 = 5” better than “two added to three is equal to five”? Fun fact: In 1557, Robert Recorde invented the equals sign, written with two parallel lines (=), because “noe 2 thynges, can be moare equalle”. (*I agrye!*)

- **The rules do the work for us**. With Roman numerals, we’re essentially recreating numbers by hand (why should VIII take so much effort to write compared to I? Oh, because 8 is larger than 1? Not a good reason!). Decimals help us “do the work” of expressing numbers, and make them easy to manipulate. So far, we’ve been doing the work of calculus ourselves: cutting a circle into rings, realizing we can unroll them, looking up the equation for area and measuring the resulting triangle. Couldn’t the rules help us here? You bet. We just need to figure them out.

- **We generalized our thinking.** “2 + 3 = 5” is really “twoness + threeness = fiveness”. It sounds weird, but we have an abstract quantity (not people, or money, or cows... just “twoness”) and we see how it’s related to other quantities. The rules of arithmetic are abstract, and it’s our job to apply them to a specific scenario.

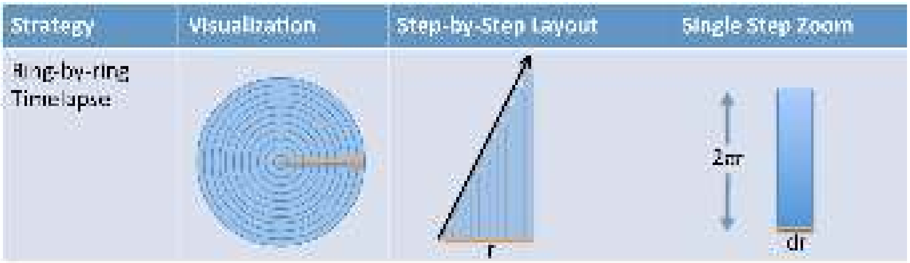
Multiplication started as a way to count groups and measure rectangular area. But when you write “\$15/hour × 2.5 hours = \$37.50” you probably aren’t thinking of “groups” of hours or getting the area of a “wage-hour” rectangle. You’re just applying arithmetic to the concepts.

In the upcoming lessons we’ll learn the official language to help us communicate our ideas and work out the rules ourselves. And once we’ve internalized the rules of calculus, we can explore patterns, whether they came from geometric shapes, business plans, or scientific theories.

CHAPTER 4

Learning The Official Terms

We've been able to describe our step-by-step process with analogies (X-Rays, Time-lapses, and rings) and diagrams:



However, this is a very elaborate way to communicate. Here's the Official Math® terms:

Intuitive Concept	Formal Name	Symbol
X-Ray (split apart)	Take the derivative (derive)	$\frac{d}{dr}$
Time-lapse (glue together)	Take the integral (integrate)	$\int$
Arrow direction	Integrate or derive "with respect to" a variable	dr implies moving along r
Arrow start/stop	Bounds or range of integration	$\int_{start}^{end}$
Slice	Integrand (shape being glued together, such as a ring)	equation, such as $2\pi r$

Let's walk through the fancy names.

The Derivative

The **derivative** is splitting a shape into sections as we move along a path (i.e., X-Raying it). Now here's the trick: although the derivative generates the entire sequence of sections (the black line), we can also extract a *single* one.

Think about a function like $f(x) = x^2$. It's a curve that describes a giant list of possibilities (1, 4, 9, 16, 25, etc.). We can graph the entire curve, sure, or examine the value of f(x) *at* a specific value, like $x = 3$.

The derivative is similar. Officially, it's the entire pattern of sections, but we can zero into a specific one by asking for the derivative *at* a certain value. (The derivative is a function, just like $f(x) = x^2$; if not otherwise specified, we're describing the entire function.)

What do we need to find the derivative? The shape to split apart, and the path to follow as we cut it up (the orange arrow). For example:

- The derivative of a circle *with respect to* the radius creates rings
- The derivative of a circle *with respect to* the perimeter creates slices
- The derivative of a circle *with respect to* the x-axis creates boards

I agree that "with respect to" sounds formal: *Honorable Grand Poombah radius, it is with respect to you that we derive.* Math is a gentleman's game, I suppose.

Taking the derivative is also called "differentiating", because we are finding the difference between successive positions as a shape grows. (As we grow the radius of a circle, the difference between the current disc and the next size up is that outer ring.)

The Integral, Arrows, and Slices

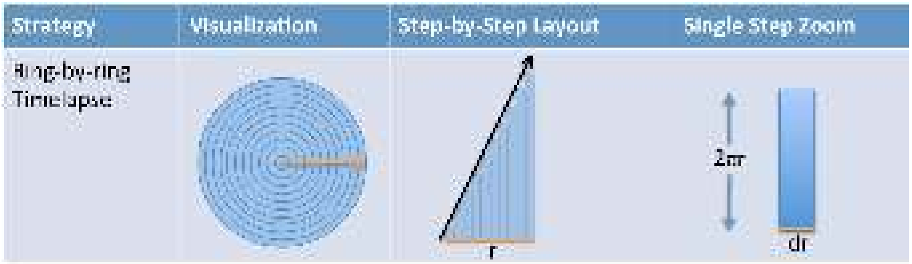
The **integral** is glueing together (time-lapsing) a group of sections and measuring the final result. For example, we glued together the rings (into a "ring triangle") and saw it accumulated to πr^2, aka the area of a circle.

Here's what we need to find the integral:

- **Which direction are we gluing the steps together?** Along the orange line (the radius, in this case)
- **When do we start and stop?** At the start and end of the arrow (we start at 0, no radius, and move to r, the full radius)
- **How big is each step?** Well... each item is a "ring". Isn't that enough?

Nope! We need to be specific. We've been saying we cut a circle into "rings" or "pizza slices" or "boards". But that's not specific enough; it's like a BBQ recipe that says "Cook meat. Flavor to taste."

Maybe an expert knows what to do, but we need more specifics. How large, exactly, is each step (technically called the "integrand")?



Ah. A few notes about the variables:

- If we are moving along the radius r, then dr is the little chunk of radius in the current step
- The height of the ring is the circumference, or $2\pi r$

There's several gotchas to keep in mind.

First, dr is its own variable, and not "d times r". It represents the tiny section of the radius present in the current step. This symbol (dr, dx, etc.) is often separated from the integrand by just a space, and it's assumed to be multiplied (written $2\pi r\ dr$).

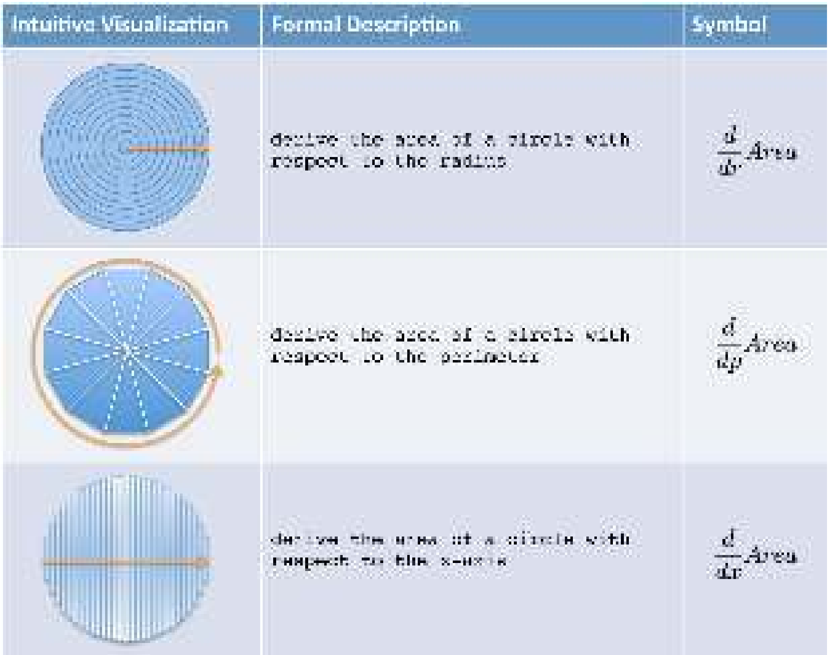
Next, if r is the only variable used in the integral, then dr is assumed to be there. So if you see $\int 2\pi r$ this still implies we're doing the full $\int 2\pi r\ dr$. (Again, if there are two variables involved, like radius and perimeter, you need to clarify which step we're using: dr or dp?)

Last, remember that r (the radius) changes as we time-lapse, starting at 0 and eventually reaching its final value. When we see r in the *context of a step*, it means "the size of the radius at the current step" and not the final value it may ultimately have.

These issues are extremely confusing. I'd prefer we use rdr to indicate an intermediate "r at the current step" instead of a general-purpose "r" that's easily confused with the max value of the radius. I can't change the symbols at this point, unfortunately.

Practicing The Lingo

Let's learn to talk like calculus natives. Here's how we can describe our X-Ray strategies:

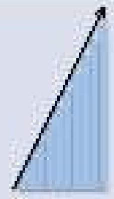
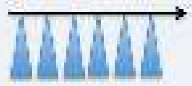
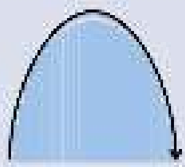


Intuitive Visualization	Formal Description	Symbol
	derive the area of a circle with respect to the radius	$\frac{d}{dr} Area$
	derive the area of a circle with respect to the perimeter	$\frac{d}{dp} Area$
	derive the area of a circle with respect to the x-axis	$\frac{d}{dx} Area$

Remember, the derivative just splits the shape into (hopefully) easy-to-measure steps, such as rings of size $2\pi r\ dr$. We broke apart our lego set and have pieces scattered on the floor. We still need an integral to glue the parts together and measure the new size. The two commands are a tag team:

- The derivative says: "Ok, I split the shape apart for you. It looks like a bunch of pieces $2\pi r$ tall and dr wide."
- The integral says: "Oh, those pieces resemble a triangle – I can measure that! The total area of that triangle is $\frac{1}{2} base \cdot height$, which works out to πr^2 in this case.".

Here's how we'd write the integrals to measure the steps we've made:

Formal Description	Symbol	Measures Total Size Of
integrate 2 * pi * r * dr from r 0 to r r	$\int_0^r 2\pi r\, dr$	
integrate [a pizza slice] from [p = min perimeter] to [p = max perimeter]	$\int_{p=min}^{p=max} (pizza\ slice)\, dp$	
integrate [a board] from [x = min value] to [x = max value]	$\int_{x=min}^{x=max} board\, dx$	

A few notes:

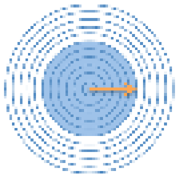
- Often, we write an integrand as an unspecified "pizza slice" or "board" (use a formal-sounding name like $s(p)$ or $b(x)$ if you like). First, we setup the integral, and then we worry about the exact formula for a board or slice.
- Because each integral represents slices from our original circle, we know they will be the same. Gluing any set of slices should always return the total area, right?
- The integral is often described as "the area under the curve". It's accurate, but shortsighted. Yes, we are gluing together the rectangular slices under the curve. But this completely overlooks the preceding X-Ray and Time-Lapse thinking. Why are we dealing with a set of slices vs. a curve in the first place? Most likely, because those slices are easier than analyzing the shape itself (how do you "directly" measure a circle?).

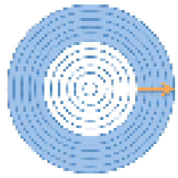
Questions

1) Can you think of another activity which is made *simpler* by shortcuts and notation, vs. written English?

2) Interested in performance? Let's drive the calculus car, even if you can't build it yet.

Question 1: How would you write the integrals that cover half of a circle?



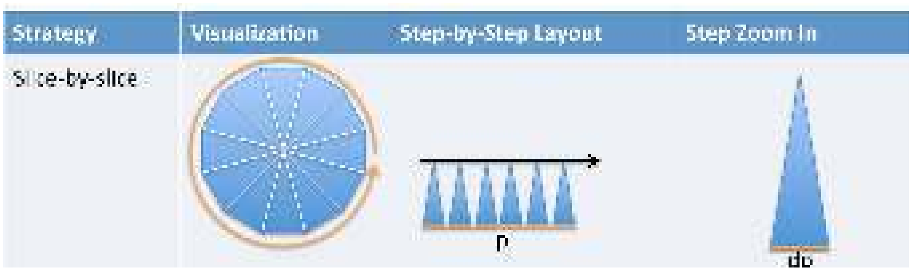


Each should would be similar to:

`integrate [size of step] from [start] to [end] with respect to [path variable]`

(Answer for the first half and the second half. This links to Wolfram Alpha, an online calculator, and we'll learn to use it later on.)

Question 2: Can you find the complete way to describe our "pizza-slice" approach?

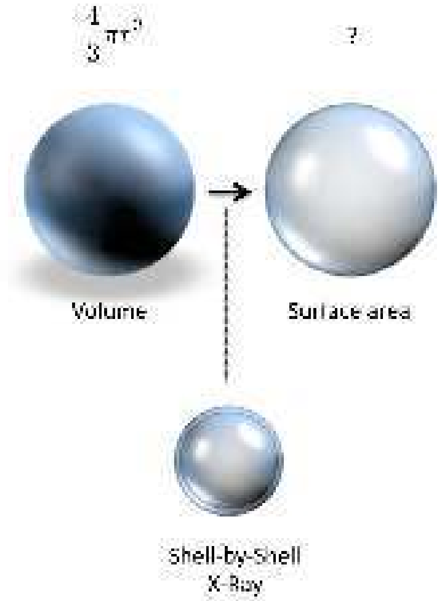


The "math command" should be something like this:

`integrate [size of step] from [start] to [end] with respect to [path variable]`

Remember that each slice is basically a triangle (so what's the area?). The slices move around the perimeter (where does it start and stop?). Have a guess for the command? Here it is, the slice-by-slice description.

Question 3: Can you figure out how to move from volume to surface area?



Assume we know the volume of a sphere is `4/3 * pi * r^3`. Think about the instructions to separate that volume into a sequence of shells. Which variable are we moving through?

`derive [equation] with respect to [path variable]`

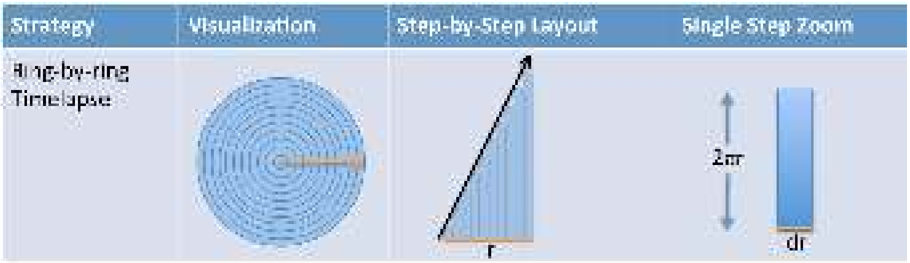
Have a guess? Great. Here's the command to turn volume into surface area.

CHAPTER 5

MUSIC FROM THE MACHINE

In the previous lessons we've gradually sharpened our intuition:

- **Appreciation:** I think it's possible to split up a circle to measure its area
- **Natural Description:** Split the circle into rings from the center outwards, like so:



- **Formal Description**: integrate 2 * pi * r * dr from r=0 to r=r
- **Performance**: (*Sigh*) I guess I'll have to start measuring the area. . .

Wait! Our formal description is precise enough that a computer can do the work for us:

Whoa! We described our thoughts well enough that a computer did the legwork.

We didn't need to manually unroll the rings, draw the triangle, and find the area (which isn't overly tough in this case, but could have been). We saw what the steps would be, wrote them down, and fed them to a computer: boomshakala, we have the result. (Just worry about the "definite integral" portion for now.)

Now, how about deriviatives, X-Raying a pattern into steps? Well, we can ask for that too:

Similar to above, the computer X-Rayed the formula for area and split it step-by-step as it moved. The result is $2\pi r$, the height of the ring at every position.

Seeing The Language In Action

Wolfram Alpha is an easy-to-use tool: the general format for calculus questions is

- `integrate [equation] from [variable=start] to [variable=end]`
- `derive [equation] with respect to [variable]`

That's a little wordy. These shortcuts are closer to the math symbols:

- `\int [equation] dr` - integrate equation (by default, assume we go from $r = 0$ to $r = r$, the max value)
- `d/dr equation` - derive equation with respect to r
- There's shortcuts for exponents (`3^2 = 9`), multiplication (`3 * r`), and roots (`sqrt(9) = 3`)

Now that we have the machine handy, let's try a few of the results we've seen so far: