$$L^2(A) \longrightarrow L^2(\hat{A})$$

$$f \mapsto \hat{f}$$

# A First Course in Harmonic Analysis

Second Edition

Universitext

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$$\| f \| = \| \hat{f} \|$$

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Anton Deitmar
Department of Mathematics
University of Exeter
Exeter, Devon EX4 4QE
UK
a.h.j.deitmar@exeter.ac.uk



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# Preface to the second edition

This book is intended as a primer in harmonic analysis at the upper undergraduate or early graduate level. All central concepts of harmonic analysis are introduced without too much technical overload. For example, the book is based entirely on the Riemann integral instead of the more demanding Lebesgue integral. Furthermore, all topological questions are dealt with purely in the context of metric spaces. It is quite surprising that this works. Indeed, it turns out that the central concepts of this beautiful and useful theory can be explained using very little technical background.

The first aim of this book is to give a lean introduction to Fourier analysis, leading up to the Poisson summation formula. The second aim is to make the reader aware of the fact that both principal incarnations of Fourier theory, the Fourier series and the Fourier transform, are special cases of a more general theory arising in the context of locally compact abelian groups. The third goal of this book is to introduce the reader to the techniques used in harmonic analysis of noncommutative groups. These techniques are explained in the context of matrix groups as a principal example.

The first part of the book deals with Fourier analysis. Chapter 1 features a basic treatment of the theory of Fourier series, culminating in $L^2$-completeness. In the second chapter this result is reformulated in terms of Hilbert spaces, the basic theory of which is presented there. Chapter 3 deals with the Fourier transform, centering on the inversion theorem and the Plancherel theorem, and combines the theory of the Fourier series and the Fourier transform in the most useful Poisson summation formula. Finally, distributions are introduced in chapter 4. Modern analysis is unthinkable without this concept that generalizes classical function spaces.

The second part of the book is devoted to the generalization of the concepts of Fourier analysis in the context of locally compact abelian groups, or LCA groups for short. In the introductory Chapter 5 the entire theory is developed in the elementary model case of a finite abelian group. The general setting is fixed in Chapter 6 by introducing the notion of LCA groups; a modest amount of topology enters at this stage. Chapter 7 deals with Pontryagin duality; the dual is shown to be an LCA group again, and the duality theorem is given.

The second part of the book concludes with Plancherel's theorem in Chapter 8. This theorem is a generalization of the completeness of the Fourier series, as well as of Plancherel's theorem for the real line.

The third part of the book is intended to provide the reader with a first impression of the world of non-commutative harmonic analysis. Chapter 9 introduces methods that are used in the analysis of matrix groups, such as the theory of the exponential series and Lie algebras. These methods are then applied in Chapter 10 to arrive at a classification of the representations of the group SU(2). In Chapter 11 we give the Peter-Weyl theorem, which generalizes the completeness of the Fourier series in the context of compact non-commutative groups and gives a decomposition of the regular representation as a direct sum of irreducibles. The theory of non-compact non-commutative groups is represented by the example of the Heisenberg group in Chapter 12. The regular representation in general decomposes as a direct integral rather than a direct sum. For the Heisenberg group this decomposition is given explicitly.

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Exeter, June 2004 Anton Deitmar

**Leitfaden**

$$
\begin{tikzcd}
 & 1 \arrow[d, dash] & & \\
 & 2 \arrow[d, dash] & & \\
 & 3 \arrow[dl, dash] \arrow[dr, dash] & & 5 \arrow[dl, dash] \\
4 & & 6 \arrow[dl, dash] \arrow[dr, dash] & \\
 & 7 \arrow[d, dash] & & 9 \arrow[d, dash] \\
 & 8 \arrow[d, dash] & & 10 \arrow[d, dash] \\
 & 12 & & 11
\end{tikzcd}
$$

**Notation** We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of natural numbers and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ for the set of natural numbers extended by zero. The set of integers is denoted by $\mathbb{Z}$, set of rational numbers by $\mathbb{Q}$, and the sets of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively.

# Contents

# Part I

# Fourier Analysis

# Chapter 1

# Fourier Series

The theory of Fourier series is concerned with the question of whether a given periodic function, such as the plot of a heartbeat or the signal of a radio pulsar, can be written as a sum of simple waves. A *simple wave* is described in mathematical terms as a function of the form $c\sin(2\pi kx)$ or $c\cos(2\pi kx)$ for an integer $k$ and a real or complex number $c$.

The formula

$$e^{2\pi ix} = \cos 2\pi x + i\sin 2\pi x$$

shows that if a function $f$ can be written as a sum of exponentials

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx},$$

for some constants $c_k$, then it also can be written as a sum of simple waves. This point of view has the advantage that it gives simpler formulas and is more suitable for generalization. Since the exponentials $e^{2\pi ikx}$ are complex-valued, it is therefore natural to consider complex-valued periodic functions.

## 1.1 Periodic Functions

A function $f : \mathbb{R} \to \mathbb{C}$ is called *periodic of period* $L > 0$ if for every $x \in \mathbb{R}$,

$$f(x + L) = f(x).$$

If $f$ is periodic of period $L$, then the function

$$F(x) \;=\; f(Lx)$$

is periodic of period 1. Moreover, since $f(x) = F(x/L)$, it suffices to consider periodic functions of period 1 only. For simplicity we will call such functions just *periodic*.

**Examples.** The functions $f(x) = \sin 2\pi x$, $f(x) = \cos 2\pi x$, and $f(x) = e^{2\pi i x}$ are periodic. Further, every given function on the half-open interval $[0, 1)$ can be extended to a periodic function in a unique way.

Recall the definition of an *inner product* $\langle ., . \rangle$ on a complex vector space $V$. This is a map from $V \times V$ to $\mathbb{C}$ satisfying

- for every $w \in V$ the map $v \mapsto \langle v, w \rangle$ is $\mathbb{C}$-linear,
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
- $\langle ., . \rangle$ is positive definite, i.e., $\langle v, v \rangle \geq 0$; and $\langle v, v \rangle = 0$ implies $v = 0$.

If $f$ and $g$ are periodic, then so is $af + bg$ for $a, b \in \mathbb{C}$, so that the set of periodic functions forms a complex vector space. We will denote by $C(\mathbb{R}/\mathbb{Z})$ the linear subspace of all *continuous* periodic functions $f : \mathbb{R} \to \mathbb{C}$. For later use we also define $C^\infty(\mathbb{R}/\mathbb{Z})$ to be the space of all infinitely differentiable periodic functions $f : \mathbb{R} \to \mathbb{C}$. For $f$ and $g$ in $C(\mathbb{R}/\mathbb{Z})$ let

$$\langle f, g \rangle \;=\; \int_0^1 f(x)\overline{g(x)}dx,$$

where the bar means complex conjugation, and the integral of a complex-valued function $h(x) = u(x) + iv(x)$ is defined by linearity, i.e.,

$$\int_0^1 h(x)dx = \int_0^1 u(x)dx + i \int_0^1 v(x)dx.$$

The reader who has up to now only seen integrals of functions from $\mathbb{R}$ to $\mathbb{R}$ should take a minute to verify that integrals of complex-valued functions satisfy the usual rules of calculus. These can be deduced from the real-valued case by splitting the function into real and imaginary part. For instance, if $f\colon [0, 1] \to \mathbb{C}$ is continuously differentiable, then $\int_0^1 f'(x)\,dx = f(1) - f(0)$.

**Lemma 1.1.1** *$\langle .,. \rangle$ defines an inner product on the vector space $C(\mathbb{R}/\mathbb{Z})$.*

**Proof:** The linearity in the first argument is a simple exercise, and so is $\langle f, g\rangle = \overline{\langle g, f\rangle}$. For the positive definiteness recall that

$$\langle f, f\rangle \;=\; \int_0^1 |f(x)|^2 dx$$

is an integral over a real-valued and nonnegative function; hence it is real and nonnegative. For the last part let $f \neq 0$ and let $g(x) = |f(x)|^2$. Then $g$ is a continuous function. Since $f \neq 0$, there is $x_0 \in [0,1]$ with $g(x_0) = \alpha > 0$. Then, since $g$ is continuous, there is $\varepsilon > 0$ such that $g(x) > \alpha/2$ for every $x \in [0,1]$ with $|x - x_0| < \varepsilon$. This implies

$$\langle f, f\rangle \;=\; \int_0^1 g(x)dx \;\geq\; \int_{|x-x_0| \,<\, \varepsilon} \frac{\alpha}{2} dx \;\geq\; \varepsilon\alpha \;>\; 0.$$

$\square$

## 1.2 Exponentials

We shall now study the periodic exponential maps in more detail. For $k \in \mathbb{Z}$ let

$$e_k(x) \;=\; e^{2\pi i k x};$$

then $e_k$ lies in $C(\mathbb{R}/\mathbb{Z})$. The inner products of the $e_k$ are given in the following lemma.

**Lemma 1.2.1** *If $k, l \in \mathbb{Z}$, then*

$$\langle e_k, e_l\rangle \;=\; \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

*In particular, it follows that the $e_k$, for varying $k$, give linearly independent vectors in the vector space $C(\mathbb{R}/\mathbb{Z})$. Finally, if*

$$f(x) \;=\; \sum_{k=-n}^{n} c_k e_k(x)$$

*for some coefficients $c_k \in \mathbb{C}$, then*

$$c_k \;=\; \langle f, e_k\rangle \quad \textit{for each } k.$$

**Proof:** If $k = l$, then

$$\langle e_k, e_l \rangle = \int_0^1 e^{2\pi ikx} e^{-2\pi ikx} dx = \int_0^1 1 \, dx = 1.$$

Now let $k \neq l$ and set $m = k - l \neq 0$; then

$$\begin{aligned} \langle e_k, e_l \rangle &= \int_0^1 e^{2\pi imx} dx \\ &= \frac{1}{2\pi im} e^{2\pi imx} \Big|_0^1 \\ &= \frac{1}{2\pi im}(1-1) = 0. \end{aligned}$$

From this we deduce the linear independence as follows. Suppose that we have

$$\lambda_{-n} e_{-n} + \lambda_{-n+1} e_{-n+1} + \cdots + \lambda_n e_n = 0$$

for some $n \in \mathbb{N}$ and coefficients $\lambda_k \in \mathbb{C}$. Then we have to show that all the coefficients $\lambda_k$ vanish. To this end let $k$ be an integer between $-n$ and $n$. Then

$$\begin{aligned} 0 &= \langle 0, e_k \rangle \\ &= \langle \lambda_{-n} e_{-n} + \cdots + \lambda_n e_n, e_k \rangle \\ &= \lambda_{-n} \langle e_{-n}, e_k \rangle + \cdots + \lambda_n \langle e_n, e_k \rangle \\ &= \lambda_k. \end{aligned}$$

Thus the $(e_k)$ are linearly independent, as claimed. In the same way we get $c_k = \langle f, e_k \rangle$ for $f$ as in the theorem. $\square$

Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and Riemann integrable on the interval $[0, 1]$. The numbers

$$c_k(f) = \langle f, e_k \rangle = \int_0^1 f(x) e^{-2\pi ikx} dx, \quad k \in \mathbb{Z},$$

are called the *Fourier coefficients* of $f$. The series

$$\sum_{k=-\infty}^{\infty} c_k(f) e^{2\pi ikx} = \sum_{k=-\infty}^{\infty} c_k(f) e_k(x),$$

i.e., the sequence of the partial sums

$$S_n(f) = \sum_{k=-n}^{n} c_k(f) e_k,$$

is called the *Fourier series* of $f$. Note that we have made no assertion on the convergence of the Fourier series so far. Indeed, it need not converge pointwise. We will show that it converges in the $L^2$-sense, a notion to be defined in the sequel.

Let $R(\mathbb{R}/\mathbb{Z})$ be the $\mathbb{C}$-vector space of all periodic functions $f : \mathbb{R} \to \mathbb{C}$ that are Riemann integrable on $[0,1]$. Since every continuous function on the interval $[0,1]$ is Riemann integrable, it follows that $C(\mathbb{R}/\mathbb{Z})$ is a subspace of $R(\mathbb{R}/\mathbb{Z})$. Note that the inner product $\langle .,. \rangle$ extends to $R(\mathbb{R}/\mathbb{Z})$, but it is no longer positive definite there (see Exercise 1.2).

For $f \in C(\mathbb{R}/\mathbb{Z})$ let

$$\|f\|_2 \;=\; \sqrt{\langle f,f\rangle}.$$

Then $\|.\|_2$ is a *norm* on the space $C(\mathbb{R}/\mathbb{Z})$; i.e.,

- it is multiplicative: $\|\lambda f\|_2 = |\lambda| \, \|f\|_2 \quad \lambda \in \mathbb{C}$,
- it is positive definite: $\|f\|_2 \geq 0$ and $\|f\|_2 = 0 \Rightarrow f = 0$,
- it satisfies the triangle inequality: $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

See Chapter 2 for a proof of this. Again the norm $\|.\|_2$ extends to $R(\mathbb{R}/\mathbb{Z})$ but loses its positive definiteness there.

## 1.3 The Bessel Inequality

The Bessel inequality gives an estimate of the sum of the square norms of the Fourier coefficients. It is of central importance in the theory of Fourier series. Its proof is based on the following lemma.

**Lemma 1.3.1** *Let $f \in R(\mathbb{R}/\mathbb{Z})$, and for $k \in \mathbb{Z}$ let $c_k = \langle f, e_k \rangle$ be its $k$th Fourier coefficient. Then for all $n \in \mathbb{N}$,*

$$\left\| f - \sum_{k=-n}^{n} c_k e_k \right\|_2^2 \;=\; \|f\|_2^2 - \sum_{k=-n}^{n} |c_k|^2.$$

**Proof:** Let $g = \sum_{k=-n}^{n} c_k e_k$. Then

$$\langle f, g \rangle \;=\; \sum_{k=-n}^{n} \overline{c_k} \, \langle f, e_k \rangle \;=\; \sum_{k=-n}^{n} \overline{c_k} c_k \;=\; \sum_{k=-n}^{n} |c_k|^2,$$

and

$$\langle g, g \rangle \;=\; \sum_{k=-n}^{n} \overline{c_k}\, \langle g, e_k \rangle \;=\; \sum_{k=-n}^{n} |c_k|^2,$$

so that

$$\begin{aligned}
\|f-g\|_2^2 &= \langle f-g, f-g \rangle \\
&= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\
&= \|f\|_2^2 - \sum_{k=-n}^{n} |c_k|^2 - \sum_{k=-n}^{n} |c_k|^2 + \sum_{k=-n}^{n} |c_k|^2 \\
&= \|f\|_2^2 - \sum_{k=-n}^{n} |c_k|^2,
\end{aligned}$$

which proves the lemma. □

**Theorem 1.3.2** *(Bessel inequality) Let $f \in R(\mathbb{R}/\mathbb{Z})$ with Fourier coefficients $(c_k)$. Then*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \;\le\; \int_0^1 |f(x)|^2 dx.$$

**Proof:** The lemma shows that for every $n \in \mathbb{N}$,

$$\sum_{k=-n}^{n} |c_k|^2 \;\le\; \|f\|_2^2 .$$

Let $n \to \infty$ to prove the theorem. □

## 1.4 Convergence in the $L^2$-Norm

We shall now introduce the notion of $L^2$-convergence, which is the appropriate notion of convergence for Fourier series. Let $f$ be in $R(\mathbb{R}/\mathbb{Z})$ and let $f_n$ be a sequence in $R(\mathbb{R}/\mathbb{Z})$. We say that the sequence $f_n$ *converges in the $L^2$-norm* to $f$ if

$$\lim_{n\to\infty} \|f - f_n\|_2 \;=\; 0.$$

Note that if a sequence $f_n$ converges to $f$ in the $L^2$-norm, then it need not converge pointwise (see Exercise 1.4). Conversely, if a sequence converges pointwise, it need not converge in the $L^2$-norm (see Exercise 1.6).

A concept of convergence that indeed does imply $L^2$-convergence is that of *uniform convergence*. Recall that a sequence of functions $f_n$ on an interval $I$ converges uniformly to a function $f$ if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$|f(x) - f_n(x)| < \varepsilon$$

for all $x \in I$. The difference between pointwise and uniform convergence lies in the fact that in the case of uniform convergence the number $n_0$ does not depend on $x$. It can be chosen uniformly for all $x \in I$.

Recall that if the sequence $f_n$ converges uniformly to $f$, and all the functions $f_n$ are continuous, then so is the function $f$.

**Examples.**

- The sequence $f_n(x) = x^n$ on the interval $I = [0, 1]$ converges pointwise, but not uniformly, to the function

  $$f(x) = \begin{cases} 0 & x < 1, \\ 1 & x = 1. \end{cases}$$

  However, on each subinterval $[0, a]$ for $a < 1$ the sequence converges uniformly to the zero function.

- Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$ for a sequence of functions $a_k(x)$, $x \in I$. Suppose there is a sequence $c_k$ of positive real numbers such that $|a_k(x)| \leq c_k$ for every $k \in \mathbb{N}$ and every $x \in I$. Suppose further that

  $$\sum_{k \in \mathbb{N}} c_k \; < \; \infty.$$

  Then it follows that the sequence $f_n$ converges uniformly to the function $f(x) = \sum_{k=1}^{\infty} a_k(x)$.

**Proposition 1.4.1** *If the sequence $f_n$ converges to $f$ uniformly on $[0, 1]$, then $f_n$ converges to $f$ in the $L^2$-norm.*

**Proof:** Let $\varepsilon > 0$. Then there is $n_0$ such that for all $n \geq n_0$,

$$|f(x) - f_n(x)| < \varepsilon \quad \text{for all } x \in [0, 1].$$

Hence for $n \geq n_0$,

$$\|f - f_n\|_2^2 \;=\; \int_0^1 |f(x) - f_n(x)|^2 dx \;<\; \varepsilon^2,$$

so that $\|f - f_n\|_2 < \varepsilon$. □

A key result of this chapter is that the Fourier series of every $f \in R(\mathbb{R}/\mathbb{Z})$ converges to $f$ in the $L^2$-norm, which we shall now prove. The idea of the proof is to find a simple class of functions for which the claim can be proved by explicit calculation of the Fourier coefficients and to then approximate a given function by those simple ones. In order to carry out these explicit calculations we shall need the following lemma.

**Lemma 1.4.2** *For $0 \leq x \leq 1$ we have*

$$\sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{k^2} \;=\; \pi^2 \left( x^2 - x + \frac{1}{6} \right).$$

Note that as a special case for $x = 0$ we get Euler's formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \;=\; \frac{\pi^2}{6}.$$

**Proof:** Let $\alpha < a < b < \beta$ be real numbers and let $f : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function. For $k \in \mathbb{R}$ let

$$F(k) \;=\; \int_a^b f(x) \sin(kx) dx.$$

*Claim:* $\lim_{|k|\to\infty} F(k) = 0$ and the convergence is uniform in $a, b \in [\alpha, \beta]$.

Proof of claim: For $t \neq 0$ we integrate by parts to get

$$F(k) \;=\; -f(x) \frac{\cos(kx)}{k} \bigg|_a^b + \frac{1}{k} \int_a^b f'(x) \cos(kx)\, dx.$$

Since $f$ and $f'$ are continuous on $[\alpha, \beta]$, there is a constant $M > 0$ such that $|f(x)| \leq M$ and $|f'(x)| \leq M$ for all $x \in [\alpha, \beta]$. This implies

$$|F(k)| \;\leq\; \frac{2M}{|k|} + \frac{M(b-a)}{|k|},$$

which proves the claim.

We employ this as follows: Let $x \in (0,1)$. Since

$$2\pi \int_{\frac{1}{2}}^{x} \cos(2\pi kt)dt = \frac{\sin(2\pi kx)}{k}$$

and

$$\sum_{k=1}^{n} \cos(2\pi kx) = \frac{\sin((2n+1)\pi x)}{2\sin(\pi x)} - \frac{1}{2},$$

we get

$$\sum_{k=1}^{n} \frac{\sin(2\pi kx)}{k} = 2\pi \int_{\frac{1}{2}}^{x} \frac{\sin((2n+1)\pi t)}{2\sin(\pi t)} dt - \pi\left(x - \frac{1}{2}\right).$$

The first summand on the right-hand side tends to zero as $n \to \infty$ by the claim. This implies that for $0 < x < 1$,

$$\sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k} = \pi\left(\frac{1}{2} - x\right),$$

and this series converges uniformly on the interval $[\delta, 1-\delta]$ for every $\delta > 0$. We now use this result to prove Lemma 1.4.2. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^2}.$$

We have just seen that the series of derivatives converges to $\pi^2(2x-1)$ and that this convergence is locally uniform, so for $0 < x < 1$ we have

$$f'(x) = \pi^2(2x-1),$$

i.e., $f(x) = \pi^2(x^2 - x) + c$. We are left to show that $c = \frac{\pi^2}{6}$. Since the series defining $f$ converges uniformly on $[0,1]$ and since $\int_0^1 \cos(2\pi kx)dx = 0$ for every $k \in \mathbb{N}$, we get

$$0 = \sum_{k=1}^{\infty} \int_0^1 \frac{\cos(2\pi kx)}{k^2} dx = \int_0^1 f(x)dx = \frac{\pi^2}{3} - \frac{\pi^2}{2} - c,$$

which implies that $c = \frac{\pi^2}{2} - \frac{\pi^2}{3} = \frac{\pi^2}{6}$. $\square$

Using this technical lemma we are now going to prove the convergence of the Fourier series for Riemannian step functions (see below) as follows.

For a subset $A$ of $[0,1]$ let $\mathbf{1}_A$ be its *characteristic function*, i.e.,

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Let $I_1, \dots, I_m$ be subintervals of $[0,1]$ which can be open or closed or half-open. A *Riemann step function* is a function of the form

$$s(x) = \sum_{j=1}^{m} \alpha_j \mathbf{1}_{I_j}(x),$$

for some coefficients $\alpha_j \in \mathbb{R}$.

Recall the definition of the Riemann integral. First, for a Riemann step function $s(x) = \sum_{j=1}^m \alpha_j \mathbf{1}_{I_j}(x)$ one defines

$$\int_0^1 s(x)dx \;=\; \sum_{j=1}^m \alpha_j \text{ length}(I_j).$$

Recall that a real-valued function $f : [0,1] \to \mathbb{R}$ is called Riemann integrable if for every $\varepsilon > 0$ there are step functions $\varphi$ and $\psi$ on $[0,1]$ such that $\varphi(x) \le f(x) \le \psi(x)$ for every $x \in [0,1]$ and

$$\int_0^1 (\psi(x) - \varphi(x))\, dx \;<\; \varepsilon.$$

As $\varepsilon$ shrinks to zero the integrals of the step functions will tend to a common limit, which is defined to be the integral of $f$. Note that as a consequence every Riemann integrable function on $[0,1]$ is bounded. A complex-valued function is called Riemann integrable if its real and imaginary parts are.

**Lemma 1.4.3** *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic and such that $f|_{[0,1]}$ is a Riemann step function. Then the Fourier series of $f$ converges to $f$ in the $L^2$-norm, i.e., the series*

$$f_n \;=\; S_n(f) \;=\; \sum_{k=-n}^{n} c_k e_k$$

*converges to $f$ in the $L^2$-norm, where for $k \in \mathbb{Z}$,*

$$c_k \;=\; \int_0^1 f(x) e^{-2\pi i k x} dx.$$

**Proof:** By Lemma 1.3.1 it suffices to show that $\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |c_k|^2$. First we consider the special case $f|_{[0,1]} = \mathbf{1}_{[0,a]}$ for some $a \in [0,1]$. The coefficients are $c_0 = a$, and

$$c_k \;=\; \int_0^a e^{-2\pi i k x} dx = \frac{i}{2\pi k}\left(e^{-2\pi i k a} - 1\right)$$

for $k \neq 0$. Thus in the latter case we have

$$|c_k|^2 \;=\; \frac{1}{4\pi^2 k^2}(e^{2\pi i k a} - 1)(e^{-2\pi i k a} - 1) \;=\; \frac{1 - \cos(2\pi k a)}{2\pi^2 k^2}.$$

Using Lemma 1.4.2 we compute

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |c_k|^2 &= a^2 + \sum_{k=1}^{\infty} \frac{1-\cos(2\pi k a)}{\pi^2 k^2} \\
&= a^2 + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} - \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k^2} \\
&= a^2 + \frac{1}{6} - \left( \frac{(1-2a)^2}{4} - \frac{1}{12} \right) \\
&= a \\
&= \int_0^1 |f(x)|^2 dx \\
&= \|f\|_2^2 .
\end{aligned}$$

Therefore, we have proved the assertion of the lemma for the function $f = \mathbf{1}_{[0,a]}$. Next we shall deduce the same result for $f = \mathbf{1}_I$, where $I$ is an arbitrary subinterval of $[0,1]$. First note that neither the Fourier coefficients nor the norm changes if we replace the closed interval by the open or half-closed interval. Next observe the behavior of the Fourier coefficients under shifts; i.e., let $c_k(f)$ denote the $k$th Fourier coefficient of $f$ and let $f^y(x) = f(x+y)$; then $f^y$ is still periodic and Riemann integrable, and

$$\begin{aligned}
c_k(f^y) &= \int_0^1 f^y(x) e^{-2\pi i k x} dx \\
&= \int_0^1 f(x+y) e^{-2\pi i k x} dx \\
&= \int_y^{1+y} f(x) e^{2\pi i k (y-x)} dx \\
&= e^{2\pi i k y} \int_0^1 f(x) e^{-2\pi i k x} dx \\
&= e^{2\pi i k y} c_k(f),
\end{aligned}$$

since it doesn't matter whether one integrates a periodic function over $[0,1]$ or over $[y, 1+y]$. This implies $|c_k(f^y)|^2 = |c_k(f)|^2$. The same argument shows that $\|f^y\|_2 = \|f\|_2$, so that the lemma now follows for $f|_{[0,1]} = \mathbf{1}_I$ for an arbitrary interval in $[0,1]$. An arbitrary step function is a linear combination of characteristic functions of intervals, so the lemma follows by linearity. □

**Theorem 1.4.4** *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and Riemann integrable on $[0,1]$. Then the Fourier series of $f$ converges to $f$ in the $L^2$-norm. If $c_k$ denotes the Fourier coefficients of $f$, then*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \;=\; \int_0^1 |f(x)|^2 dx.$$

The theorem in particular implies that the sequence $c_k$ tends to zero as $|k| \to \infty$. This assertion is also known as the *Riemann-Lebesgue Lemma.*

**Proof:** Let $f = u + iv$ be the decomposition of $f$ into real and imaginary parts. The partial sums of the Fourier series for $f$ satisfy $S_n(f) = S_n(u) + iS_n(v)$, so if the Fourier series of $u$ and $v$ converge in the $L^2$-norm to $u$ and $v$, then the claim follows for $f$. To prove the theorem it thus suffices to consider the case where $f$ is real-valued. Since, furthermore, integrable functions are bounded, we can multiply $f$ by a positive scalar, so we may assume that $|f(x)| \le 1$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$. Since $f$ is Riemann integrable, there are step functions $\varphi, \psi$ on $[0,1]$ such that

$$-1 \;\le\; \varphi \;\le\; f \;\le\; \psi \;\le\; 1$$

and

$$\int_0^1 (\psi(x) - \varphi(x))dx \;\le\; \frac{\varepsilon^2}{8}.$$

Let $g = f - \varphi$ then $g \ge 0$ and

$$|g|^2 \;\le\; |\psi - \varphi|^2 \;\le\; 2(\psi - \varphi),$$

so that

$$\int_0^1 |g(x)|^2 dx \;\le\; 2\int_0^1 (\psi(x) - \varphi(x))dx \;\le\; \frac{\varepsilon^2}{4}.$$

For the partial sums $S_n$ we have

$$S_n(f) \;=\; S_n(\varphi) + S_n(g).$$

By Lemma 1.4.3 there is $n_0 \ge 0$ such that for $n \ge n_0$,

$$\|\varphi - S_n(\varphi)\|_2 \;\le\; \frac{\varepsilon}{2}.$$